Probability of Sufficiency of Solvency II Reserve Risk Margins: Practical Approximations

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Origin of this work …

The new IFRS 17 standard (May 2017, effective on 1 Jan 2022) brings ***one specific requirement to disclose confidence level of reserve risk margins …***

No specific guidance from IASB on the calculation of confidence level of reserve risk margins. This research work:

- looks for practical ways of implementing this new requirement; and
- proposes a distribution-free approach to estimating IFRS 17 confidence level of reserve risk margins in a 'standard formula' style.

Outline

1. Background

- **Reserve risk margin:** reserve risk and its profile characterisation;
- **Probability of Sufficiency (PoS)** as a measure of IFRS 17 Confidence Level;
- **Distribution-free approximation of PoS:** information-based approach

2. Distribution-free approximation of PoS

- **'Statistical DNA'** of risk profile;
- Two types of distribution-free approximations: **Cornish-Fisher (C-F)** and **Bohman-Esscher (B-E);**
- Practical implementations: standalone reserving class and portfolio of multiple reserving classes.

Reserve risk and its profile

- **Risk (SII) -** a possibility of having adverse performance outcomes (e.g. insurance, investment, reserving) that result in 'low capital performance' and/or capital consumption.

- **Reserve risk -** provisions for past exposures will be inadequate to meet the ultimate costs when the business is run off to extinction. Significant for non-life insurers, especially for long-tail business.

- **Reserve risk carrier -** reserve value. The distribution of *reserve risk carrier* characterises the reserve risk and is called *reserve risk profile*. Differentiated by type of business:
    - personal vs. commercial;
    - short-tail vs. long-tail (duration and convexity).

Reserve risk margin (1)

Market-consistent valuation of liabilities:

Booked Reserve Provision = Best Estimate (BE) + Risk Margin (RM)

- **Complete market**: fully hedgeable liabilities, replicating portfolio (mark-to-market) approach - *RM is implicitly embedded in the risk-adjusted expected value of future liabilities under a unique martingale measure*

- **Incomplete market**: non-hedgeable insurance risks, non-uniqueness of martingale measure

    - Optimal hedging approaches: *implicit allowance for RM in market-consistent value*

    - Cost of Capital (CoC) approach: *explicit calculation of RM*

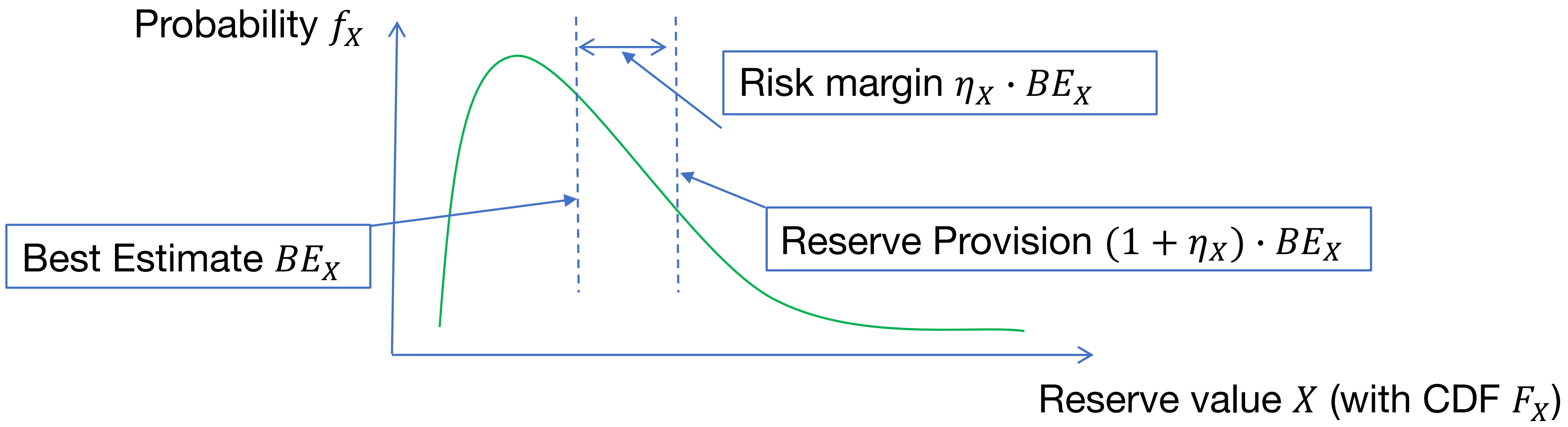
Reserve risk margin (2)

RM plays the role of safety load reflecting the uncertainty in reserve central estimate, whereas the actual volatility of reserving process (reserve risk) is fully absorbed by capital requirements:

$$\mathbb{E}\left[\left(BE-\widehat{BE}\right)^2\right]=\mathbb{E}\left[\left(\left(BE-\mathbb{E}\left[\widehat{BE}\right]\right)+\left(\mathbb{E}\left[\widehat{BE}\right]-\widehat{BE}\right)\right)^2\right]=\text{bias}^2+\text{Var}\left[\widehat{BE}\right]$$

$$=\underbrace{\text{bias}^2}_{\text{SII Pillar II}}+\mathbb{E}\underbrace{\left[\text{Var}\left[\widehat{BE}\mid\mathscr{F}\right]\right]}_{\text{SII SCR}}+\text{Var}\underbrace{\left[\mathbb{E}\left[\widehat{BE}\mid\mathscr{F}\right]\right]}_{\text{RM}}$$

PoS as a measure of confidence level (1)



- **Confidence level** of reserve risk margin per unit of best estimate (often pre-calculated and provided by reserving/capital actuaries) *is defined by the probability level* of booked provision on the reserve distribution curve. It is called *Probability of Sufficiency (PoS)* and equal to:

$$PoS = F_X\left[\left(1+\eta_X\right) \cdot BE_X\right] = \mathbb{P}\left[X \leq \left(1+\eta_X\right) \cdot BE_X\right]$$

- **Examples of PoS use:** in Australia the non-life technical provisions are required by APRA to be set at the minimum level which is the greater of the two values:
  - 75th percentile of the reserve distribution providing 75% of PoS; and
  - best estimate plus one half of a standard deviation of the reserve.

PoS as a measure of confidence level (2)

- **PoS itself is a measure of prudence** in liability valuation:
  - PoS below 50% - technical provisions (TPs) are set below the central estimate (under-reserved position);
  - PoS of 50% to 60% - TPs are set approximately at the level of central estimate (weak prudence);
  - PoS of around 75% - likely (i.e. up to 1-in-4 years) reserve deteriorations above the central estimate are fully absorbed by TPs (adequate prudence); and
  - PoS above 75% - TPs could also absorb some of unlikely reserve deteriorations above the central estimate (strong prudence).

PoS in a log-normal world (1)

- **What if the reserve is log-normally distributed? … i.e.**

$X \sim LN(\mu, \sigma^2)$ **and** $BE_X = e^{\mu + \frac{1}{2}\sigma^2}, \;\; \sigma^2 = \ln\left(1 + CoV_X^2\right)$ **and** $\frac{X}{BE_X} \sim LN(-\frac{1}{2}\sigma^2, \sigma^2)$

$$PoS = F_X\left[\left(1+\eta_X\right)\cdot BE_X\right] = \mathbb{P}\left[\frac{X}{BE_X} \leq \left(1+\eta_X\right)\right]$$

$$= \Phi\left(\frac{\ln\left(1+\eta_X\right) + \frac{1}{2}\sigma^2}{\sigma}\right)$$

$$= \Phi\left(\frac{\ln\left(\left(1+\eta_X\right)\cdot\sqrt{1+CoV_X^2}\right)}{\sqrt{\ln\left(1+CoV_X^2\right)}}\right)$$

- **… and the problem is solved, but …**

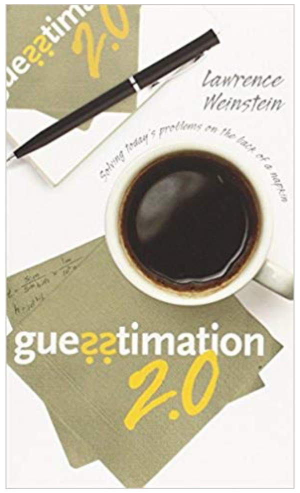
PoS in a log-normal world (2)

- **Log-normality does not cover the whole range of practically feasible reserve risk profiles,** i.e. $CoV \leq 50\%$ and disproportionally higher (lower) skewness
  - How about $CoV = 20\%$ and skewness of 1.0 (or 0.4) or equivalently 5 (or 2), when expressed per unit of CoV (i.e. *Skewness*-to-*CoV* (*SC*) ratio)?
  - With log-normality we could only achieve the SC ratio in the range of 3 to 3.25 for $CoV \leq 50\%$
- Is there a way of **estimating PoS using only** the reserve risk profile's characteristics like **CoV and skewness** without knowing the parametric structure of reserve distribution?

PoS approximation (1)

Rather inspired by the idea of **being able to**

1. **… solve any world's problem on the back of a cocktail napkin …** *If all French baguettes sold in Paris last year were placed end-to-end, what distance would they cover?*



2. … but also, most importantly, **find a way of gaining information-based insight** into PoS level:

$$PoS = \mathbb{P}\left[X \leq \left(1 + \eta_X\right) \cdot BE_X\right] = \alpha$$

Information Characterisation (Information Theory)

vs.

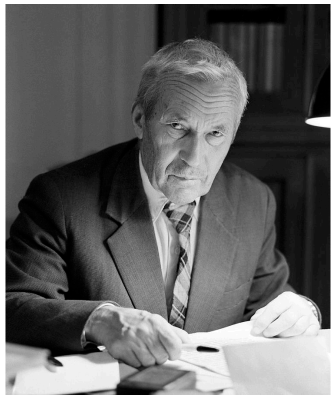
Frequentist Statistician's Approach (Fitting/Inverting Distributions)

Randomness Criteria and Information Theory

Some relevant history…

F. P. Cantelli’s early probabilistic problems:

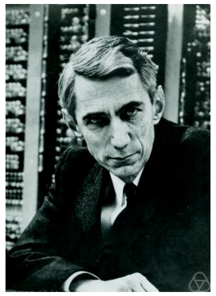
- “Una teoria astratta del calcolo delle probabilità”, *Giornale dell’Istituto Italiano degli Attuari*. Vol. III - 2, 1932
- Meeting with Kolmogorov, Rimini (Italy), 1932



A.N. Kolomogorov’s foundations of the probability theory:

- “Sulla determinazione empirico di una legge di distribuzione”, *Giornale dell’Istituto Italiano degli Attuari*. Vol. IV, 1933
- “*Grundbegriffe der Wahrscheinlichkeitrechnung*”, Ergebnisse der Mathematik. Berlin, 1933.

Significance of Shannon’s ‘Information Theory’ 1948-1956



PoS approximation (2)

- **Frequentist's inversion of distribution** (or finding $\mathrm{VaR}_\alpha(X)$) requires perfect knowledge of the distribution of X.

- **The information-based insight** can be gained by using the centralised and normalised copy of X, $\widetilde{X} = \frac{X - BE_X}{BE_X \cdot CoV_X}$:

$$X = BE_X \cdot \left(1 + CoV_X \cdot \widetilde{X}\right);$$

$$\mathrm{VaR}_\alpha(X) = BE_X \cdot \left(1 + \eta_X\right) = BE_X \cdot \left(1 + CoV_X \cdot \mathrm{VaR}_\alpha\left(\widetilde{X}\right)\right);$$

$$\mathrm{VaR}_\alpha\left(\widetilde{X}\right) = \frac{\eta_X}{CoV_X}.$$

PoS approximation (3)

$\mathrm{VaR}_\alpha\left(\widetilde{X}\right) = \frac{\eta_X}{CoV_X}$ — it carries ***statistical DNA*** of reserve risk profile:

- *CoV, Skewness-to-CoV (SC) ratio* and *Kurtosis-to-CoV^{2}ratio* are the key components of statistical DNA of reserve distribution, as they:
  - provide a vital topology (SSP topology) for characterising distributions; and
  - explain quantiles of $\widetilde{X}$ well.

Let's have a thought experiment to demonstrate the role 'information' can play using an example from Number Theory … your attention is much appreciated!

A few additional features of $\mathrm{VaR}_\alpha\left(\widetilde{X}\right)$

- It is dimensionless, or simply invariant w.r.t. reserve risk volume;
- It also represents the so-called *Coefficient of Riskiness (CoR)* - the quantile's distance from the mean measured in units of standard deviation of the underlying reserve risk profile
  - Chebyshev's inequality is used to estimate the upper bound of CoR (David Ingram, WillisRe)

$$\mathbb{P}\left[\left|\widetilde{X}\right| \geq k\right] \leq \frac{1}{k^2} \qquad \left(\textbf{or} \leq \frac{4}{9k^2}\right)$$

Chebyshev's inequality and its improved variants (Petunin's inequality) **are crude estimates.**

Proposed distribution-free approximations provide a better solution!

Single Shape Parameter distributions (1)

- Most distributions commonly used in insurance for reserving and loss modelling are:
    - two-parameter distributions with ***scale*** and ***shape*** parameters being separated;
    - the shape of the distribution is fully explained by its shape parameter.
- They are called **Single Shape Parameter (SSP) distributions**
    - Examples include: *Gamma, Inverse-Gaussian (Wald), Log-Normal, Dagum, Suzuki, Exponentiated-Exponential (Verhulst), Inverse-Gamma (Vinci), Birnbaum-Saunders, Exponentiated-Frechet, Reciprocal Wald and Log-Logistic.*

Single Shape Parameter distributions (2)

SSP distributions can be split into categories:

- **Moderately skewed** distributions ($1.5 < SC \leq 3$)
  - *Gamma, Inverse-Gaussian (Wald);*
- **Significantly skewed** distributions ($3 < SC \leq 4$)
  - *Log-Normal, Suzuki, Exponentiated-Exponential (Verhulst) and Dagum*
- **Extremely skewed** distributions ($SC > 4$)
  - *Inverse-Gamma (Vinci), Birnbaum-Saunders, Exponentiated-Frechet, Reciprocal Wald and Log-Logistic.*

SSP characterisation of reserves (1)

Table 2: *SC and KCsq ratios for the four parametric distributions.*

Parametric distribution	**SC ratio as a function of CoV**	**KCsq ratio as a function of $\mathbf{CoV}^2$**
Gamma	2	6
Inverse-Gaussian (Wald)	3	15
Log-Normal	$3 + CoV^2 \in (3, 4),\ CoV < 100\%$	$16 + 15CoV^2 + 6CoV^4 + CoV^6 > 16$
Inverse-Gamma (Vinci)	$\frac{4}{1-CoV^2} > 4,\ CoV < 100\%$	$\frac{30\left(1-\frac{1}{5}CoV^2\right)}{\left(1-CoV^2\right)\left(1-2CoV^2\right)} > 30,\ CoV < 70\%$

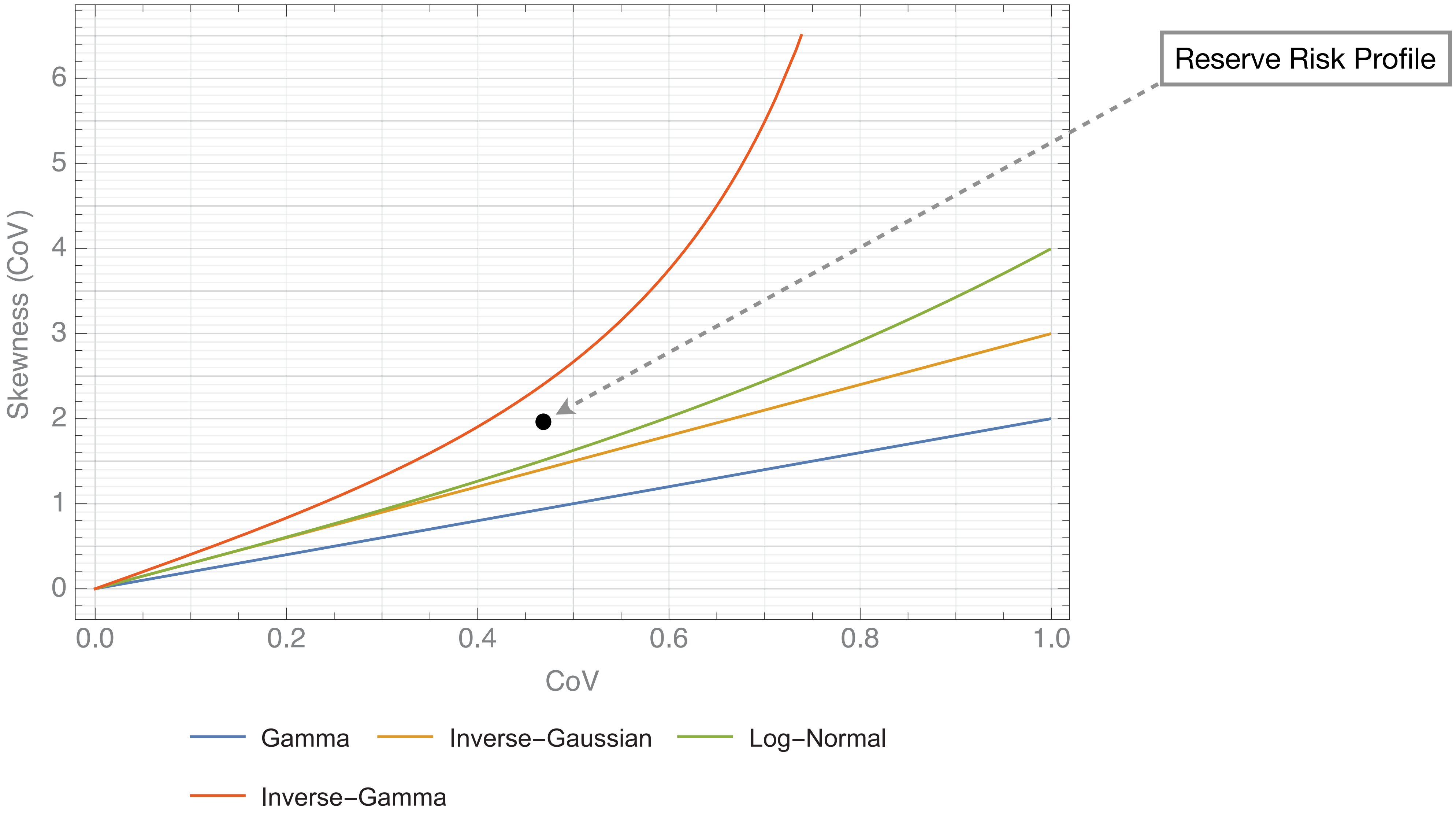
Table 1: *Differentiation of reserve risk profile by type of reserve class.*

Type of reserving class				
Duration	**CoV range**	**Skewness (SC ratio)**	**Parametric distribution(s)**	**Example of reserving class**
Short tail	10%-12%	1.9 to 2.1	Gamma	Motor (ex Bodily Injury)
Short tail	12%-16%	2.0 to 3.0	Gamma, Inverse-Gaussian (Wald)	Home
Short tail	10%-16%	2.9 to 3.1	Inverse-Gaussian (Wald), Log-Normal	Comm Property/Fire, Comm Accident
Long tail	12%-25%	3.0 to 3.5	Log-Normal	Motor Bodily Injury, Marine
Long tail	18%-50%	3.0 to 4.0	Log-Normal, Inverse-Gamma (Vinci)	Workers Comp, Prof Liab, Comm Liab
Long tail	25%-70%	> 4	Inverse-Gamma (Vinci)	Asbestos and other long tail books

SSP characterisation of reserves (2)

SSP topology - Skewness vs. CoV:

Figure 1: *Skewness as a function of CoV for the four parametric distributions.*



Distribution-free approximations of PoS (1)

- **Bohman-Esscher (B-E)** based on transformed Gamma distribution

$$\tilde{\alpha} \approx \mathrm{Gamma}_{s,1}\left(s+\sqrt{s}\cdot q\right) = \frac{1}{\Gamma(s)}\int_{0}^{s+\sqrt{s}\cdot q} y^{s-1}e^{-y}\,dy$$

where $s=\frac{4}{\gamma_X^2};\quad \gamma_X = SC_X\cdot CoV_X;\quad q=\frac{\eta_X}{CoV_X}.$

- **Cornish-Fisher (C-F)** - includes the case of Normal-Power (N-P)

$$\mathrm{VaR}_\alpha\left(\widetilde{X}\right) \approx z_\alpha + \gamma_X\cdot\frac{z_\alpha^2-1}{6} + C_1\left(\iota_X\cdot\frac{z_\alpha^3-3z_\alpha}{24} - \gamma_X^2\cdot\frac{2z_\alpha^3-5z_\alpha}{36}\right)$$

$$+C_2\left(\gamma_X^3\cdot\frac{12z_\alpha^4-53z_\alpha^2+17}{324} - \gamma_X\cdot\iota_X\cdot\frac{z_\alpha^4-5z_\alpha^2+2}{24}\right)$$

Distribution-free approximations of PoS (2)

- **The quality of both B-E and N-P approximations is generally quite good** (Dal Moro and Krvavych (2017)) when compared to other alternative approximations of distribution's VaR like *higher-order Cornish-Fisher approximations, Haldane method, Wilson-Hilferty method* and *GPD/Hill method*.

- **When compared to the N-P approximation, in most practical cases the B-E approximation is of the same or better quality.**

- **The two approximations can be easily implemented in Excel environment**

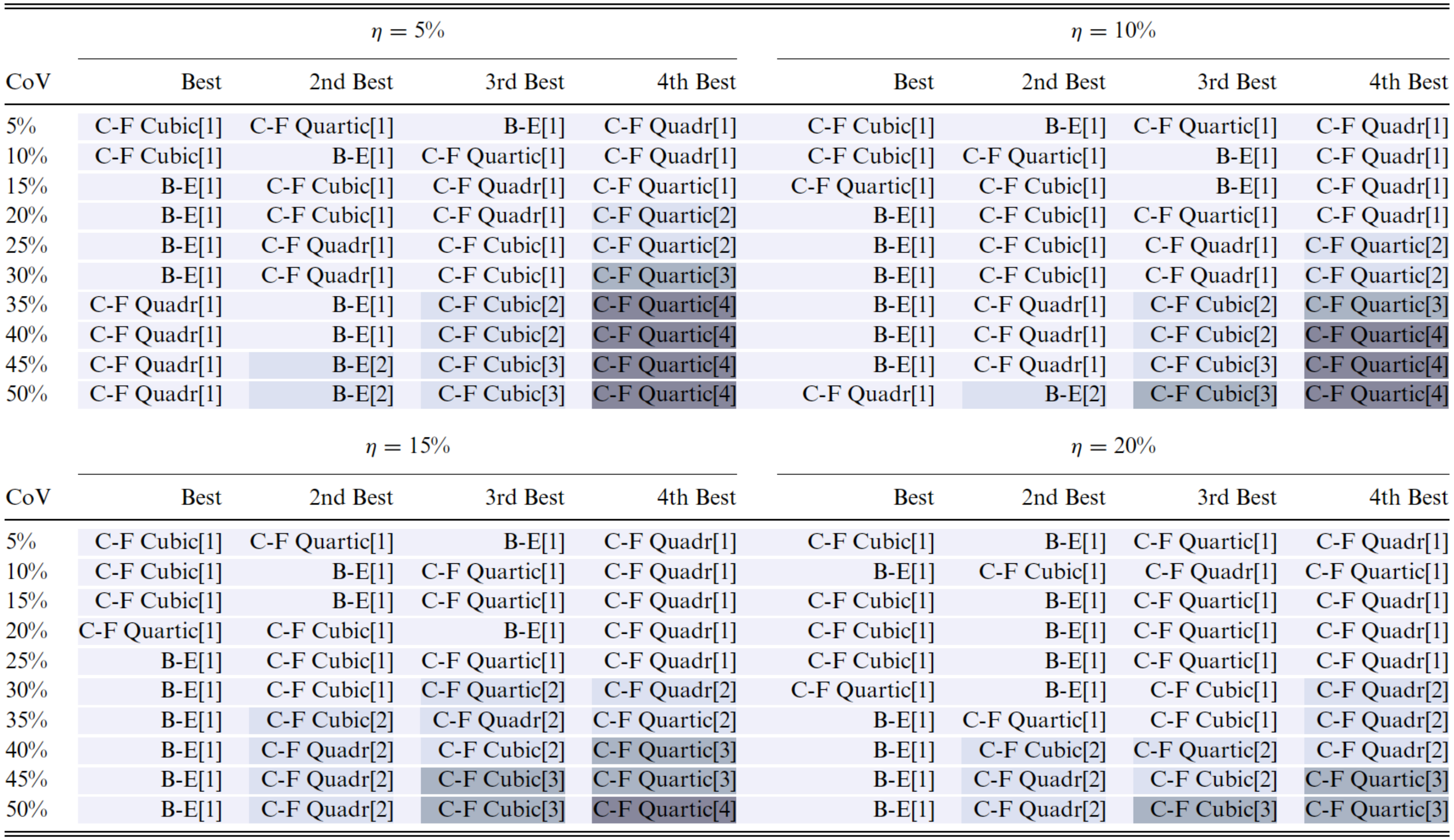
    - B-E approximation can be computed using Excel's Gamma distribution function `GAMMA.DIST (x, alpha, beta, cumulative)`, i.e. using the following formulaic expression **`GAMMA.DIST(s + Sqrt(s)·q, s, 1, 1)`**

Distribution-free approximations of PoS (3)

Relative error of PoS approximation

Band	Bracketed Number	Colour Key
$\lvert\Delta\rvert \leqslant 1\%$	1	░
$1\% < \lvert\Delta\rvert \leqslant 2.5\%$	2	▒
$2.5\% < \lvert\Delta\rvert \leqslant 5\%$	3	▓
$\lvert\Delta\rvert > 5\%$	4	█

ANALYSIS OF QUALITY OF APPROXIMATIONS: SUMMARY RESULTS FOR ESTIMATING LOG-NORMAL DISTRIBUTION.



	$\eta = 5\%$				$\eta = 10\%$			
CoV	Best	2nd Best	3rd Best	4th Best	Best	2nd Best	3rd Best	4th Best
5%	C-F Cubic[1]	C-F Quartic[1]	B-E[1]	C-F Quadr[1]	C-F Cubic[1]	B-E[1]	C-F Quartic[1]	C-F Quadr[1]
10%	C-F Cubic[1]	B-E[1]	C-F Quartic[1]	C-F Quadr[1]	C-F Cubic[1]	C-F Quartic[1]	B-E[1]	C-F Quadr[1]
15%	B-E[1]	C-F Cubic[1]	C-F Quadr[1]	C-F Quartic[1]	C-F Quartic[1]	C-F Cubic[1]	B-E[1]	C-F Quadr[1]
20%	B-E[1]	C-F Cubic[1]	C-F Quadr[1]	C-F Quartic[2]	B-E[1]	C-F Cubic[1]	C-F Quartic[1]	C-F Quadr[1]
25%	B-E[1]	C-F Quadr[1]	C-F Cubic[1]	C-F Quartic[2]	B-E[1]	C-F Cubic[1]	C-F Quadr[1]	C-F Quartic[2]
30%	B-E[1]	C-F Quadr[1]	C-F Cubic[1]	C-F Quartic[3]	B-E[1]	C-F Cubic[1]	C-F Quadr[1]	C-F Quartic[2]
35%	C-F Quadr[1]	B-E[1]	C-F Cubic[2]	C-F Quartic[4]	B-E[1]	C-F Quadr[1]	C-F Cubic[2]	C-F Quartic[3]
40%	C-F Quadr[1]	B-E[1]	C-F Cubic[2]	C-F Quartic[4]	B-E[1]	C-F Quadr[1]	C-F Cubic[2]	C-F Quartic[4]
45%	C-F Quadr[1]	B-E[2]	C-F Cubic[3]	C-F Quartic[4]	B-E[1]	C-F Quadr[1]	C-F Cubic[3]	C-F Quartic[4]
50%	C-F Quadr[1]	B-E[2]	C-F Cubic[3]	C-F Quartic[4]	C-F Quadr[1]	B-E[2]	C-F Cubic[3]	C-F Quartic[4]

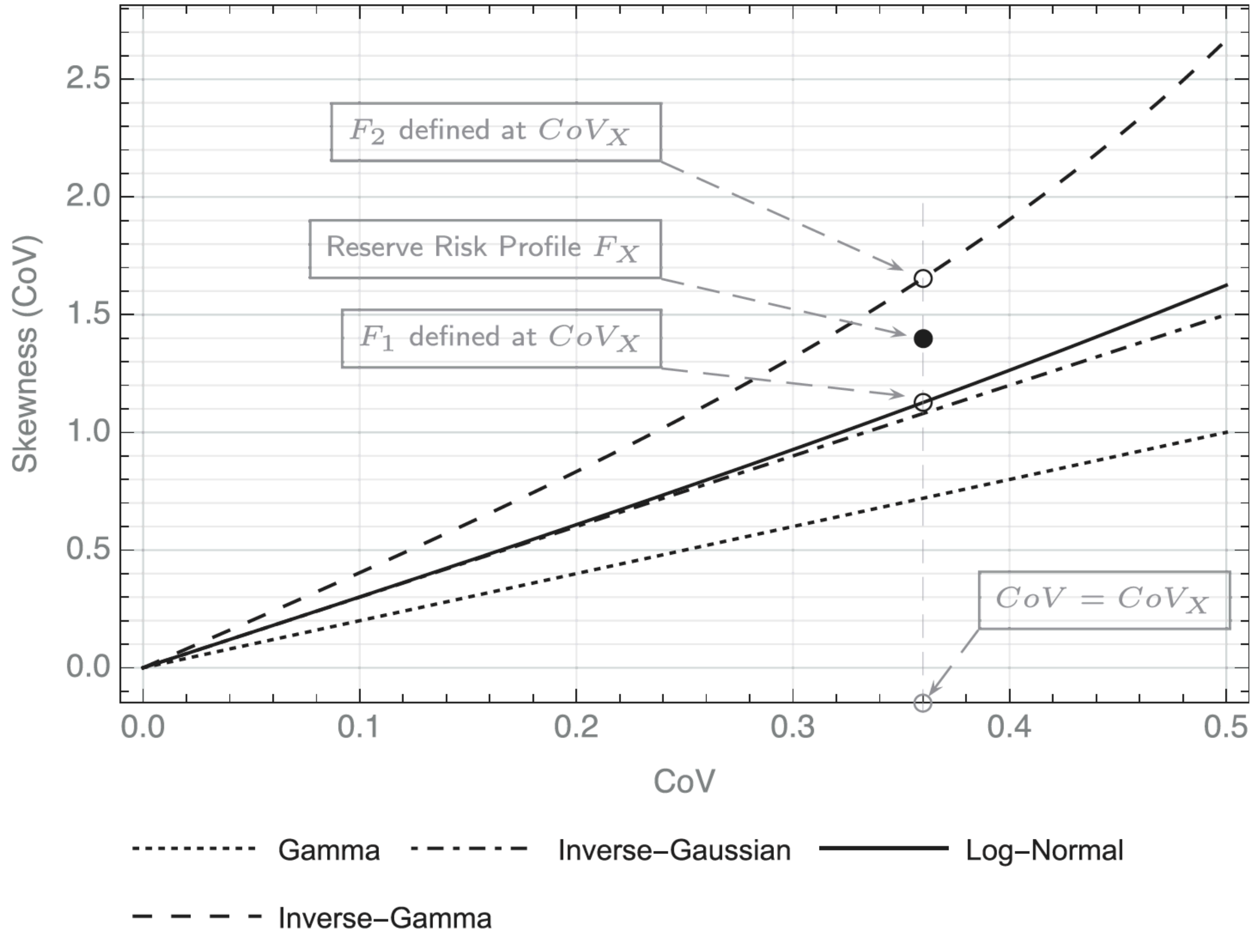
	$\eta = 15\%$				$\eta = 20\%$			
CoV	Best	2nd Best	3rd Best	4th Best	Best	2nd Best	3rd Best	4th Best
5%	C-F Cubic[1]	C-F Quartic[1]	B-E[1]	C-F Quadr[1]	C-F Cubic[1]	B-E[1]	C-F Quartic[1]	C-F Quadr[1]
10%	C-F Cubic[1]	B-E[1]	C-F Quartic[1]	C-F Quadr[1]	B-E[1]	C-F Cubic[1]	C-F Quadr[1]	C-F Quartic[1]
15%	C-F Cubic[1]	B-E[1]	C-F Quartic[1]	C-F Quadr[1]	C-F Cubic[1]	B-E[1]	C-F Quartic[1]	C-F Quadr[1]
20%	C-F Quartic[1]	C-F Cubic[1]	B-E[1]	C-F Quadr[1]	C-F Cubic[1]	B-E[1]	C-F Quartic[1]	C-F Quadr[1]
25%	B-E[1]	C-F Cubic[1]	C-F Quartic[1]	C-F Quadr[1]	C-F Cubic[1]	B-E[1]	C-F Quartic[1]	C-F Quadr[1]
30%	B-E[1]	C-F Cubic[1]	C-F Quartic[2]	C-F Quadr[2]	C-F Quartic[1]	B-E[1]	C-F Cubic[1]	C-F Quadr[2]
35%	B-E[1]	C-F Cubic[2]	C-F Quadr[2]	C-F Quartic[2]	B-E[1]	C-F Quartic[1]	C-F Cubic[1]	C-F Quadr[2]
40%	B-E[1]	C-F Quadr[2]	C-F Cubic[2]	C-F Quartic[3]	B-E[1]	C-F Cubic[2]	C-F Quartic[2]	C-F Quadr[2]
45%	B-E[1]	C-F Quadr[2]	C-F Cubic[3]	C-F Quartic[3]	B-E[1]	C-F Quadr[2]	C-F Cubic[2]	C-F Quartic[3]
50%	B-E[1]	C-F Quadr[2]	C-F Cubic[3]	C-F Quartic[4]	B-E[1]	C-F Quadr[2]	C-F Cubic[3]	C-F Quartic[3]

PoS approximation - further refinement

- Calculate correction factors when on one of the parametric curves in the SSP topology (ratio of exact-to-approximate)

- Locate the reserve risk profile between two adjacent (closest) SSP curves …

- Ultimate correction factor is then interpolated from the correction factors of the two adjacent SSP curves



PoS approximation - portfolio of classes (1)

$$X_{\Sigma} = \sum_{i=1}^{m} X_i$$

Reserve of each standalone class is approximated by a ***Fleishman's polynomial*** of standard normal (calibrated to unit variance and given skew):

$$X_i \approx BE_i \cdot \left(1 + CoV_i \cdot P_3(Z_i)\right),$$

$$P_3(Z_i) = a_i Z_i + b_i \left(Z_i^2 - 1\right) + c_i Z_i^3$$

The idea is to:

- impose a Gaussian dependence structure with standard normals serving as meta-distributions;
- calculate CoV, skewness (and kurtosis) at the portfolio level; and
- use them in the PoS approximations derived earlier for a standalone class.

PoS approximation - portfolio of classes (2)

Portfolio variance

$$\mathrm{Var}[X_\Sigma] = \mathbb{E}\left[\left(\sum_{i=1}^{m} \sigma_i \cdot P_3(Z_i)\right)^2\right]$$

$$= \sum_{i=1}^{m} \sigma_i^2 + 2\sum_{ij} \sigma_i \sigma_j \cdot \mathbb{E}\left[P_3(Z_i)P_3(Z_j)\right],$$

where $\sigma_i = BE_i \cdot CoV_i; \quad \mathbb{E}\left[P_3^2(Z_i)\right] = 1.$

$$\mathbb{E}\left[P_3(Z_i)P_3(Z_j)\right] = \rho_{ij}\left(a_i a_j + 2b_i b_j \rho_{ij} + 3\left(a_i c_j + a_j c_i\right)\right.$$

$$\left. +3c_i c_j\left(3 + 2\rho_{ij}^2\right)\right)$$

PoS approximation - portfolio of classes (3)

Portfolio third central moment

$$\mathbb{E}\left[\left(X_\Sigma - BE_\Sigma\right)^3\right] = \mathbb{E}\left[\left(\sum_{i=1}^m \sigma_i \cdot P_3(Z_i)\right)^3\right]$$

$$= \sum_{i=1}^m \sigma_i^3 \cdot \gamma_i + 3\sum_{ij} \sigma_i^2 \sigma_j \cdot \mathbb{E}\left[P_3(Z_i)^2 P_3(Z_j)\right]$$

$$+6\sum_{ijk} \sigma_i \sigma_j \sigma_k \cdot \mathbb{E}\left[P_3(Z_i) P_3(Z_j) P_3(Z_k)\right].$$

and for $c_i = 0$

$$\mathbb{E}\left[P_3(Z_i)^2 P_3(Z_j)\right] = 2\rho_{ij}\left(2a_i a_j b_i + \left(a_i^2 + 4b_i^2\right) b_j \rho_{ij}\right)$$

$$\mathbb{E}\left[P_3(Z_i) P_3(Z_j) P_3(Z_k)\right] = 2\left(a_j a_k b_i \rho_{ij} \rho_{ik} + a_j a_i b_k \rho_{jk} \rho_{ik} + a_i a_k b_j \rho_{ij} \rho_{jk}\right)$$

$$+8 b_i b_j b_k \rho_{ij} \rho_{ik} \rho_{jk}$$

R implementations (ChainLadder)

The latest **ChainLadder 0.2.10 package** (Markus Gesmann) includes new functions:

- `QuantileIFRS17`
- `quantile.MackChainLadder`

Please visit CRAN for more details:

https://cran.r-project.org/web/packages/ChainLadder/index.html

Thank you!

References

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