Fat-tailed Distributions for Investment Variables

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Outline

- Use "Wilkie model"
- Fit model for each variable assuming normal residuals.
- Take these residuals and fit different distributions.

Introduction

- Possible distributions

1. Normal
2. Laplace (double exponential)
3. Skew Laplace
4. Hyperbolic
5. Skew hyperbolic

- All examples of "conical distributions"

Conical Distributions

- Take some conic section:
  parabola, hyperbola, two straight lines
- Limit to those with full range for x, $-\infty$ to $+\infty$
  omit circle, ellipse, etc
- Arrange as $y = g(x)$
- Choose that part with $y < 0$
- Put $h(x) = \exp(y)$
- Take density $f(x) = k.h(x)$
- Find k so that $\int f(x).dx = 1$
  i.e. find $1 / k = \int h(x).dx$

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Conical Distributions

- Parabola with nose at (0, 0) axis vertical

- This gives Normal distribution

  $y = -ax^2$

  $f(x) = k.\exp(-ax^2)$

  $\mu = 0 \qquad 1/a = 2\sigma^2 \qquad 1/k = \sigma\sqrt{(2\pi)}$

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Conical Distributions

- Two straight lines, symmetric, crossing at (0, 0)

  Laplace, two symmetric exponentials

  $$f(x) = \alpha.\exp(-\text{abs}(\alpha x)) / 2$$

  $k = \alpha/2$

  $0 < \alpha$

  Often parameterised with $\lambda = 1/\alpha$

Conical Distributions

- Two straight lines, skewed, crossing at (0, 0)
- **Skew Laplace**, two different exponentials, meeting at x = 0

  $f(x) = k.\exp(\alpha(1+\rho)x) \quad x < 0$

  $\quad\quad = k.\exp(-\alpha(1-\rho)x) \quad x > 0$

  $k = \alpha(1 - \rho^2)/2$

  $0 < \alpha \qquad -1 < \rho < +1$

- Could be parameterised with λ_1, λ_2

Conical Distributions

- Hyperbola with main axis vertical

  asymptotes crossing at (0, 0) symmetric

- Gives **hyperbolic**

  $f(x) = k.\exp(-\alpha\delta.\sqrt{1 + (x/\delta)^2})$

  $1/k = 2\delta.K_1(\alpha\delta)$

  $K_1(.)$ is one of the Bessel functions

  $0 < \delta \qquad 0 < \alpha$

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Conical Distributions

- **Skew hyperbola gives skew hyperbolic**

  $$f(x) = k.\exp\left(-\alpha\delta.\sqrt{1 + (x/\delta)^2} + \rho x/\delta\right)$$

- **Put** $\gamma = \alpha.\sqrt{(1 - \rho^2)}$

  $$1/k = 2\alpha.\delta.K_1(\gamma\delta) / \gamma$$

  $K_1(.)$ **as before a Bessel function**

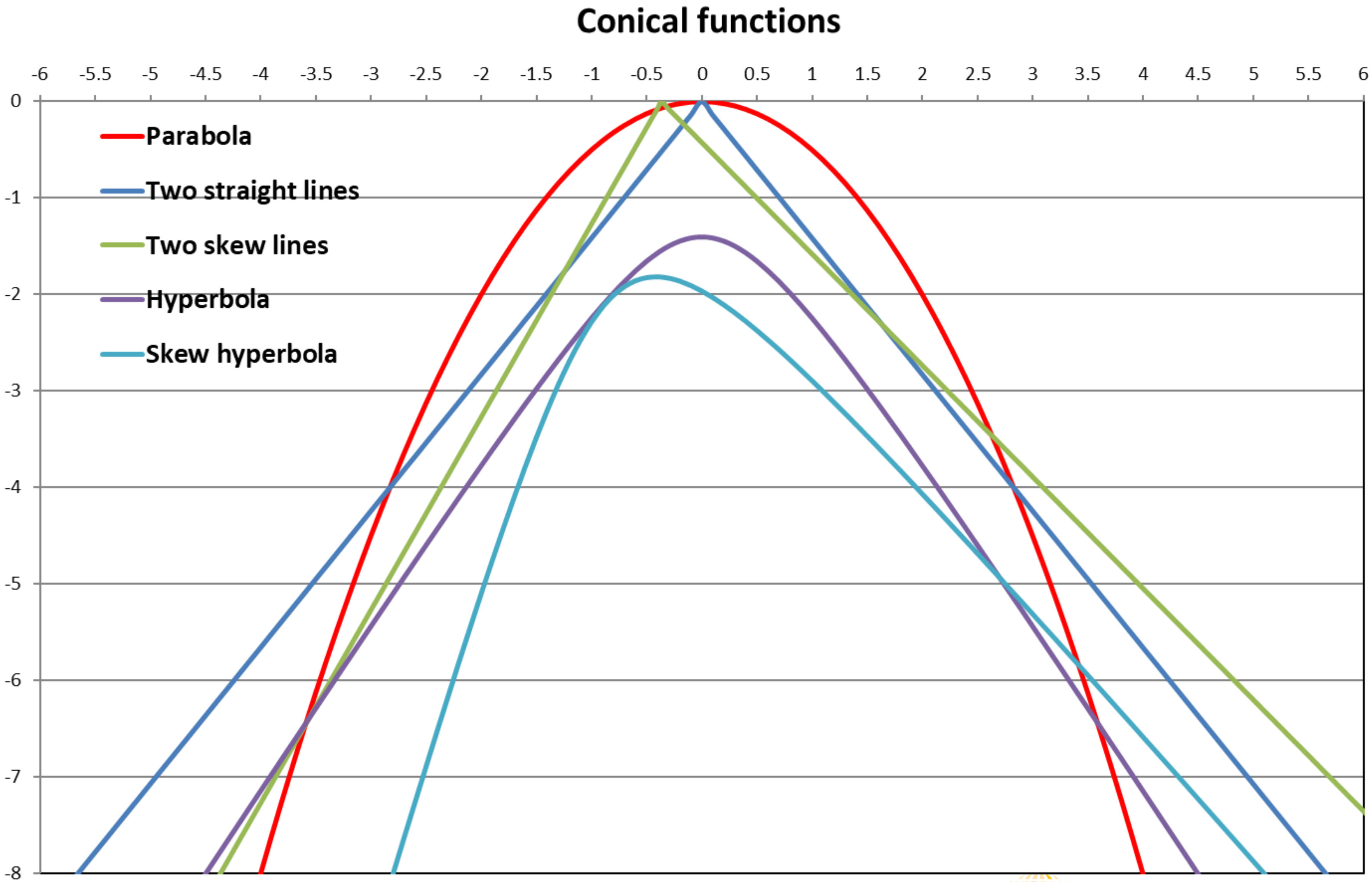
  $0 < \delta \qquad 0 < \alpha \qquad -1 < \rho < +1$

- **Often parameterised differently**

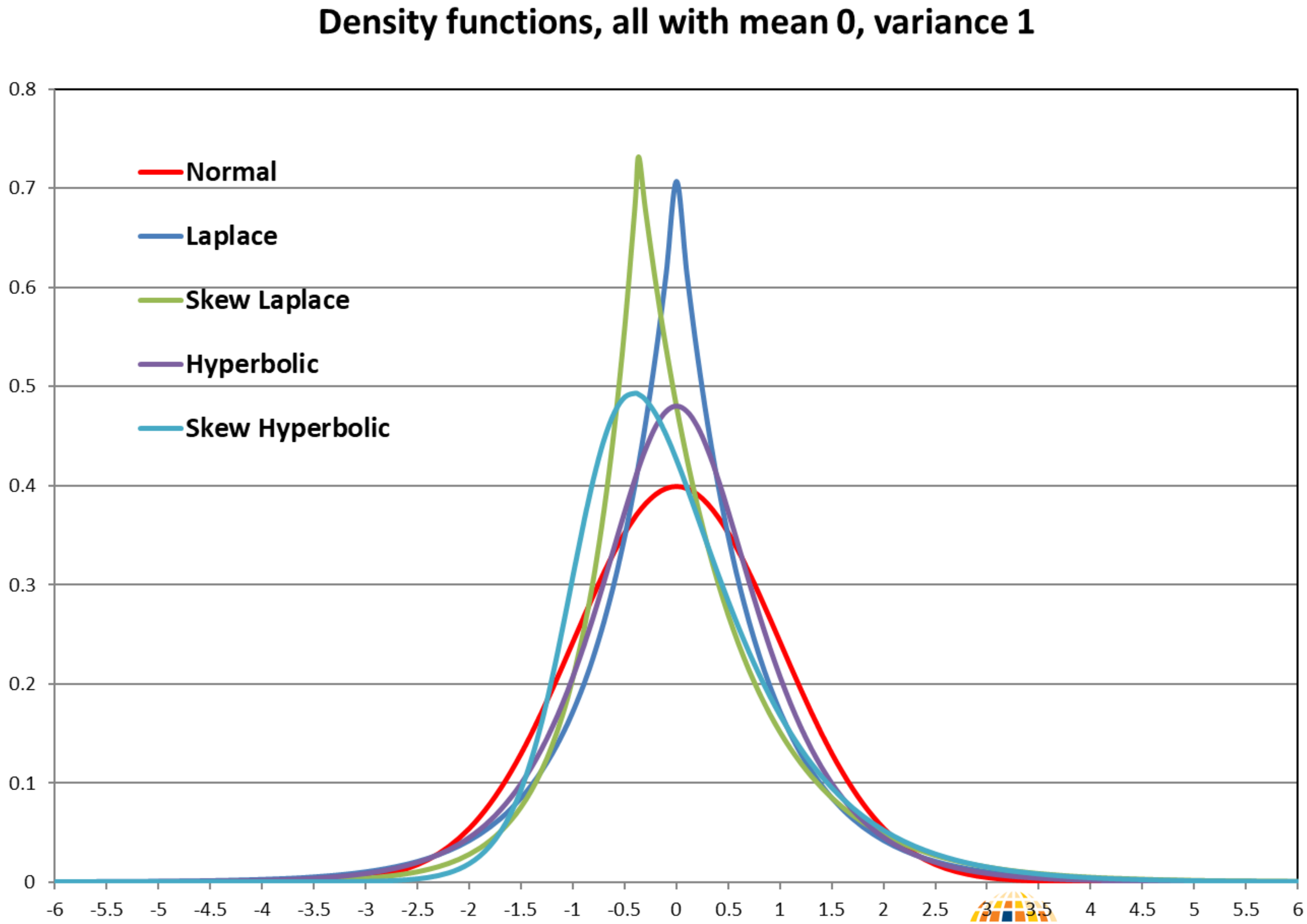
Conical Distributions

- All can be offset from (0, 0) to (μ, 0)
- For symmetric versions this gives mean μ
- For skew versions, mean depends on parameters
- There is a scale factor for each, e.g. σ, 1/*α*, δ

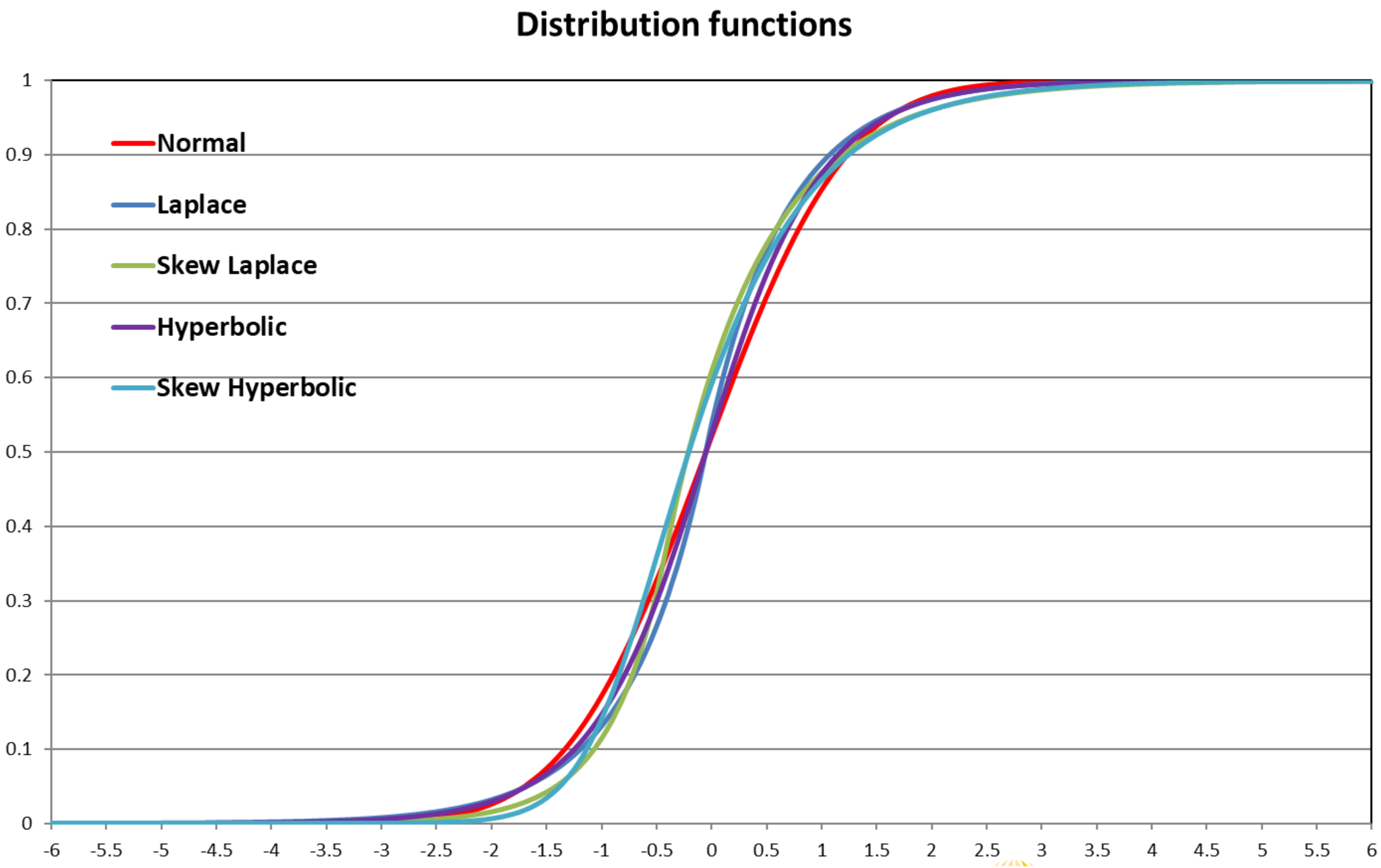
Conical Distributions



Conical Distributions



Conical Distributions



Conical Distributions

- **Limiting versions:**

  If $\rho = -1$ or $\rho = +1$

  One of the straight lines (or asymptotes)

  becomes the vertical axis.

  No longer full range of x

- **For hyperbola:**

  if $\alpha = 0$ we get two straight lines

  if $\alpha = \infty$ we get parabola

The Wilkie Model – Principal Variables

- Consumer prices index, Q
- Wages index, W
- Share dividends, D
- Share dividend yield, $Y = D/P$
- Share earnings, E
- Cover, $V = E/D$
- Multiple, $M = P/E$ ratio
- Long interest rate, C
- Short term interest rate, B
- Real yield on index-linked bonds, R

The Wilkie Model - Residuals

- **For each series *x*, *xZ* is the standardised residual, i.e.**

  Normal (0, 1)

- **Consider first Skewness and Kurtosis**

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The Wilkie Model - Residuals

Series	Skewness	Kurtosis	p(J-B)	Normal?
QZ	1.31	6.39	0.0	No
WZ	0.39	3.92	0.0526	Possibly
YZ	0.35	3.57	0.1998	Yes
DZ	–0.74	4.22	0.0006	No
EZ	–0.29	10.30	0.0	No
VZ	0.37	3.67	0.3152	Yes
MZ	–0.64	4.57	0.0081	No
CZ	–0.75	5.50	0.0	No
BZ	–3.61	25.98	0.0	No
RZ	–0.34	2.43	0.5422	Yes

The Wilkie Model

Compare log-likelihood for other distribution with log-likelihood for Normal

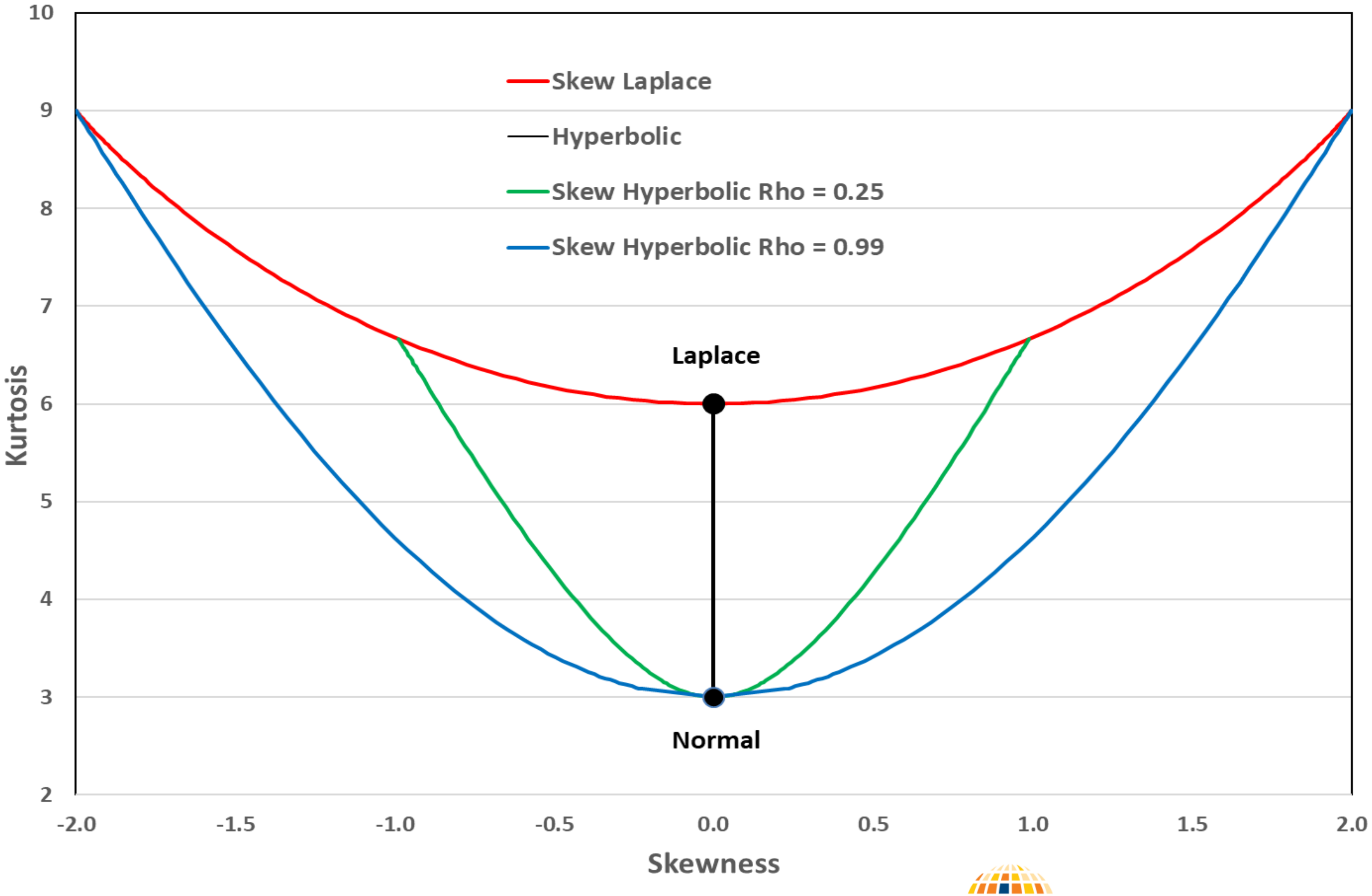
Comparison of log-likelihoods for different dist.

Series	L–N	SL–N	H–N	SH–N
QZ	10.12	10.65	10.15	10.71
WZ	1.17	1.23	1.96	2.97
YZ	–3.23	–1.12	0.30	0.84
DZ	2.28	4.98	2.69	4.98
EZ	8.51	8.65	8.73	8.78
VZ	0.25	0.54	0.95	1.34
MZ	0.03	2.02	1.31	2.04
CZ	4.88	6.31	5.06	6.48
BZ	25.85	26.22	25.87	26.22
RZ	–1.69	–1.25	–0.01	0.65

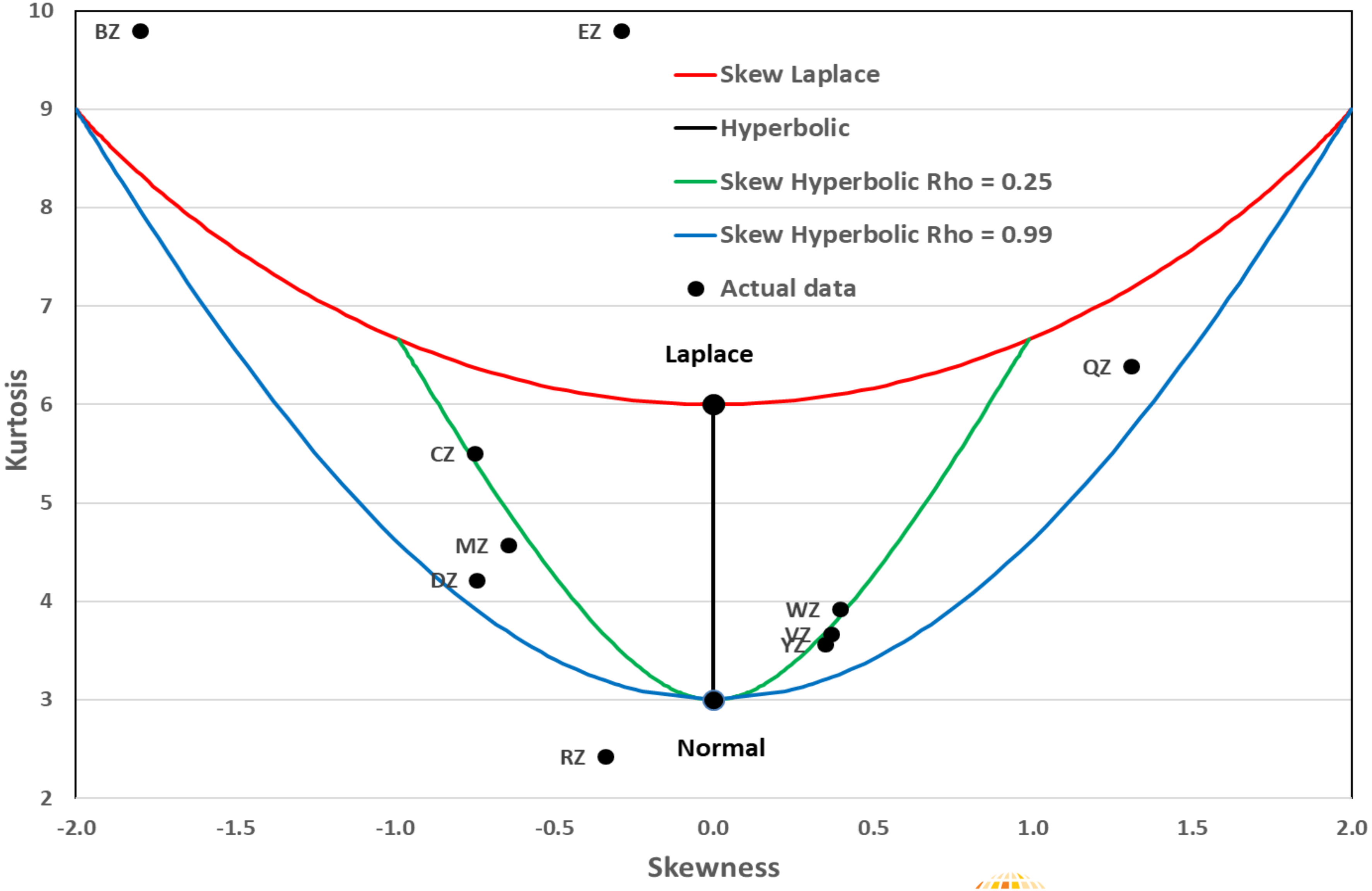
Skewness-Kurtosis Diagram

- Skewness-Kurtosis (S - K) diagram
- Normal (S, K) = (0, 3)
- Laplace (S, K) = (0, 6)
- Skew Laplace varies with ρ on a line
  from (–2, 9) to (0, 6) to (+2, 9)
- Hyperbolic, S = 0, K varies with α
  on a line (0, 3) to (0, 6)
- Skew Hyperbolic varies with α and ρ
  within a 'triangular' area

Skewness-Kurtosis (S-K) Diagram



S-K Diagram with points for Actual Values



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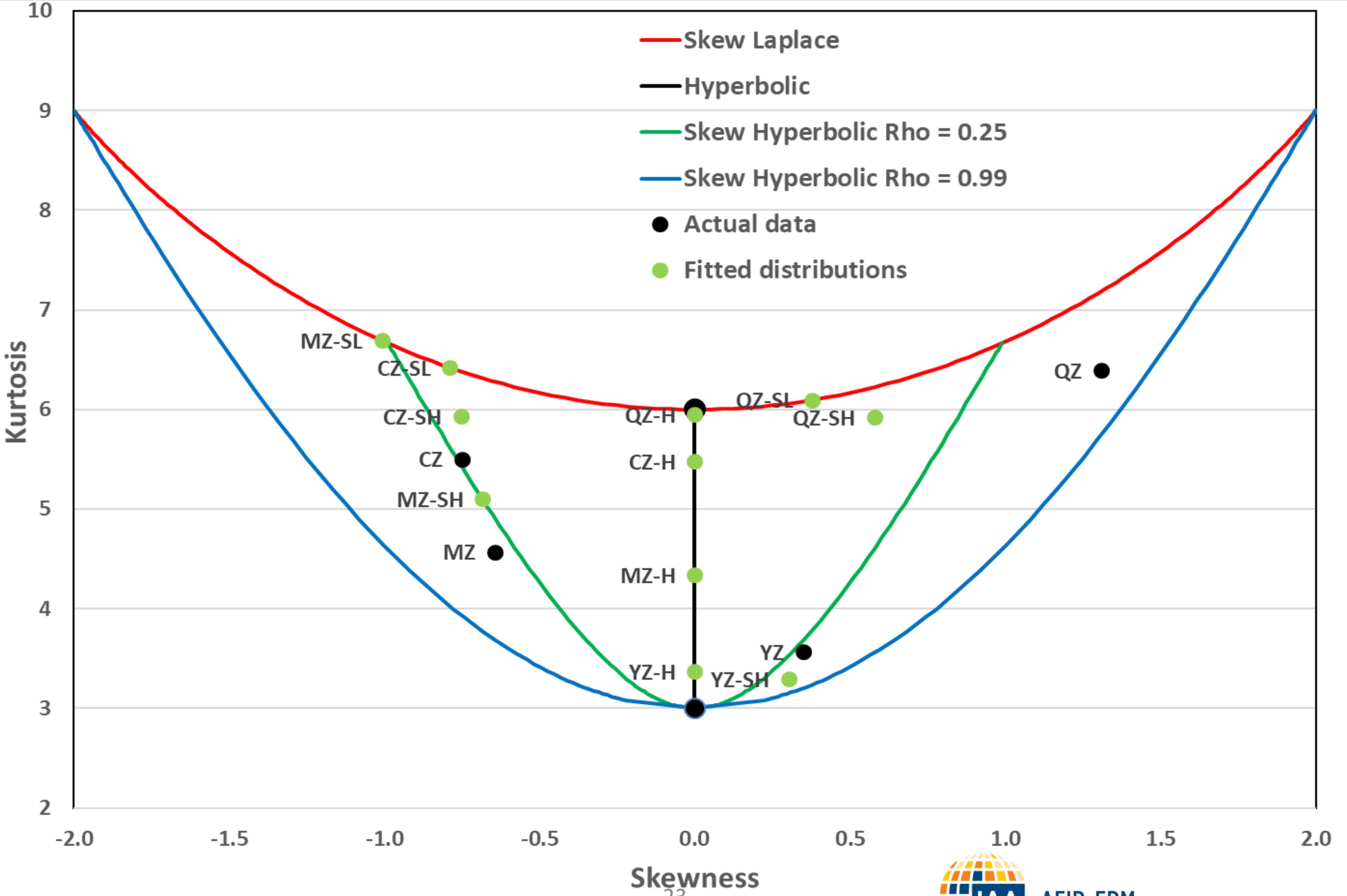
S-K Diagram -points for Actual and Fitted Values



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Comparison of log-likelihoods for different dist.

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WZ				
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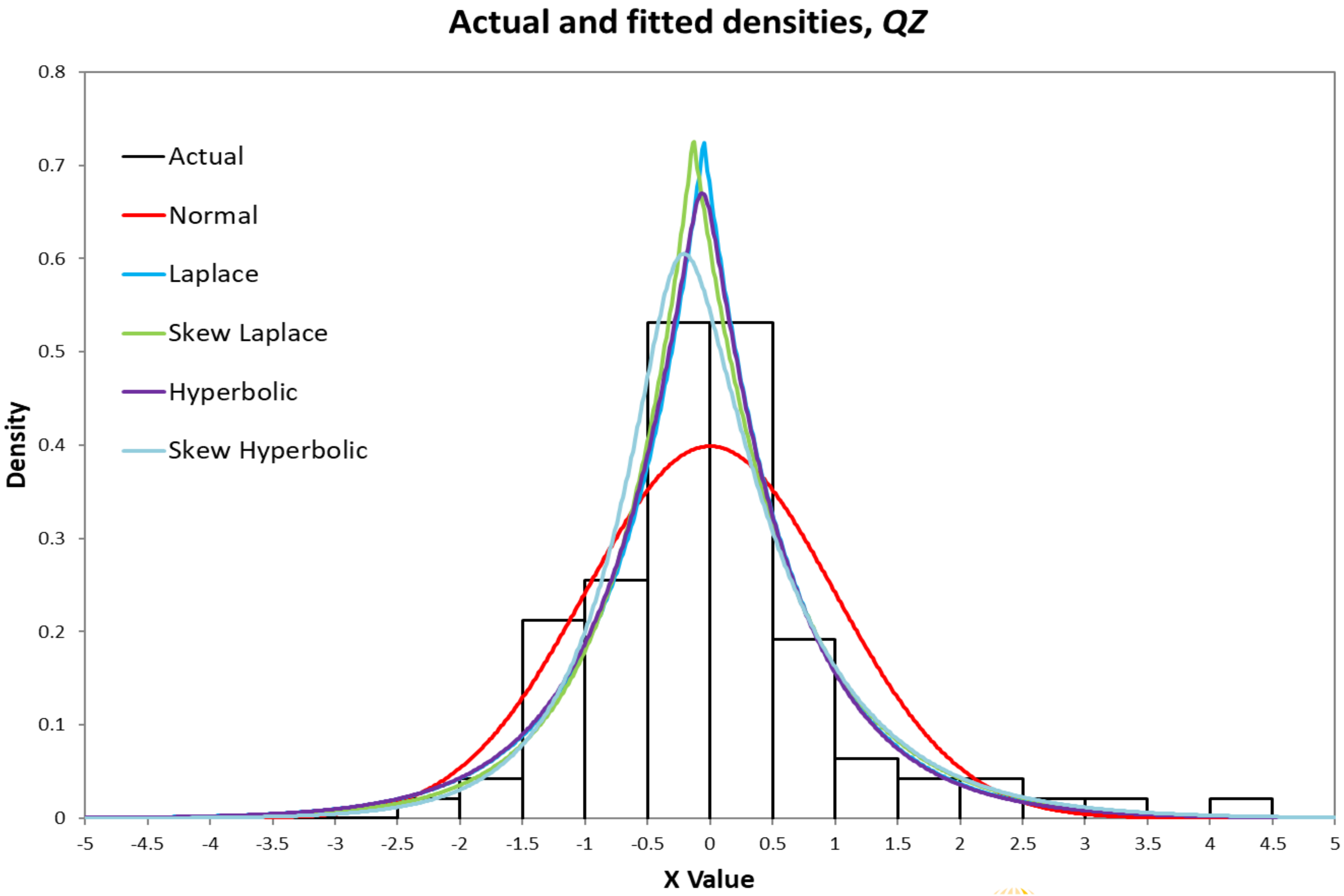
CZ	4.88	6.31	5.06	6.48
BZ				
RZ				

Conclusions

- **Normal:** ***YZ, VZ, RZ***
- **Laplace:** ***BZ, EZ***
- **Skew Laplace:** ***DZ***
- **Hyperbolic:** **none**
- **Skew Hyperbolic:** ***QZ, WZ, CZ, MZ***

The Wilkie Model – Retail Prices



The Wilkie Model – Simulations

- **Simulations**

  Normal as usual (Marsaglia's method)

  Laplace by inversion

  Hyperbolic acceptance/rejection method

- **Principal variables:**

  Q, W, D, P, E **index-type**

  Y, V, M, C, B, R **ratio-type**

- **Total Return indices, including income**

  PT, CT, BT, RT **index-type**

The Wilkie Model – Simulations

Compound continuous rate of total return

$$GQL(t) = \{\ln(Q(t)) - \ln(Q(0)\} / t$$
$$= \{QL(t) - QL(0)\} / t$$

Nominal

$$GWL(\mathrm{t}) = \{WL(t) - WL(0)\} / t$$

'Real' rate

$$GWLR(\mathrm{t}) = GWL(t) - GQL(t)$$

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The Wilkie Model – Simulations

- 1,000,000 simulations for 50 years.
- Very large amounts of output.

- Criteria based on quantiles $Q(a)$

  $\frac{1}{2}\,(Q(1 - a) - Q(a))$ / Standard Deviation

  a = 5% gives 90% spread, C 90%

  a = 0.5% gives 99% spread, C 99%

  Normal C 90% = 1.64 and C 99% = 2.58

The Wilkie Model – Simulations

Retail Prices, *GQL*(*t*)

Results with Skew Hyperbolic innovations

Term, *t*	1	2	5	10	20	50
Skewness	0.59	0.44	0.28	0.19	0.13	0.08
Kurtosis	5.99	4.76	3.69	3.32	3.16	3.06
C 90%	1.62	1.62	1.63	1.64	1.64	1.64
C 99%	3.19	2.99	2.77	2.67	2.62	2.60

- Kurtosis reduces with *t* so does Skewness
- C 90% same as Normal
- C 99% larger than Normal, reduces with *t*

The Wilkie Model – Simulations

- **Most variables** like **Normal** with varying Kurtosis in year 1

- But **Long-term interest rates**, *C*, with *GCTR*(*t*) and **Short-term interest rates**, *B*, with *GBTR*(*t*) **different**, because basic model *mixes Normal and Lognormal* so even with Normal innovations there is very high Kurtosis.

Future Work

- **Still to do:**

- **Re-estimate all the parameters of all the variables with an appropriate new distribution**

- **For Retail Prices, Skew Laplace becomes best**

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- **Wilkie, A.D. & Şahin Ş. (submitted to *AAS*). Yet more on a stochastic economic model: Part 6A, Allowing for parameter uncertainty with a hypermodel, and also fat-tailed Innovations for the retail prices model**

- **Wilkie, A.D. & Şahin Ş. (in draft). Yet more on a stochastic economic model: Part 6B, Investigating distributions for residuals using the Normal parameters for the skeleton**