Fair valuation of insurance liability cash-flow streams in continuous time

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Introduction

- The fundamental question in finance and insurance: how should we price contingent claims?
- Actuarial valuation is based on the idea of diversification. The actuarial value of the claim is defined as the expectation under the real-world measure plus an additional risk loading to cover any non-diversifiable risk,
- Risk-neutral (arbitrage-free) valuation is based on the idea of replicating investment portfolios. The arbitrage-free value of the claim is defined as the expectation under the risk-neutral measure and it coincides with the market price of the replicating portfolio,
- Market-consistent valuation extends the arbitrage-free pricing operator with the unique risk-neutral measure to the general set of non-hedgeable claims where we have to choose the pricing objective and the pricing martingale measure. Recently formalized as an extension of the notion of cash-invariance to all hedgeable claims,
- Fair valuation – a valuation which is actuarial and market-consistent.

Solvency II Directive

- Insurance liabilities should be valued in a market-consistent way,
- The market-consistent value of an insurance liability is defined as the sum of two elements:

$$MC\ Value \;=\; Best\ Estimate \;+\; Risk\ Margin,$$

- The best estimate corresponds to the expected value of the future cash flows weighted with probabilities and discounted with the risk-free interest rate,
- The risk margin equals the amount of the funds necessary to support the insurance obligations over their lifetime,
- Market-consistency means that the model price of an asset traded actively in the financial market must coincide with the price of this asset observed in the market,
- The best estimate is only sufficient for hedgeable risks. The risk margin has to be added for non-hedgeable risks in order to protect the insurer from adverse deviations.

IFRS 17

- The general model measures a group of the insurance contracts as the sum of:
- An unbiased and probability-weighted estimate of the future cash flows,
- A discount adjustment to the present value to reflect the time value of money and financial risks (the discount rates may include illiquidity premiums),
- A risk adjustment for non-financial risks,
- The risk adjustment is the compensation that the entity requires for bearing the uncertainty about the amount and timing of cash flows that arise from non-financial risks,
- In contrast to Solvency II, the risk adjustment is an entity-specific perception and results from indifference pricing of the variable cash flows - it is not the value that reflects transfer to market participants.

Overview

- We consider a general financial and insurance model with hedegable and non-hedgeable financial risks and non-hedgeable insurance risk,
- We start with a one-period hedge-based valuation where an optimal dynamic hedging portfolio for the liability is set up with traded assets and the non-hedgeable part of the liability is valuated via a subjective actuarial valuation,
- We define a multi-period valuation operator by backward iterations of the one-period valuation operator[1],
- We investigate the continuous-time limit of the multi-period, discrete-time iterations and derive a partial differential equation for the continuous-time valuation operator which satisfies the limit,
- Our continuous-time valuation operator is fair and decomposes into the best estimate of liability and the risk margin – it is agrees with the Solvency II and IFRS 17 valuation rules,
- The dynamic hedging strategy associated with the continuous-time valuation operator is established.

[1] One-period and multi-period hedge-based valuations with static hedging strategies were introduced by J. Dhaene and K. Brigou.

The financial market

- The risk-free asset:

$$\frac{dR(t)}{R(t)} = r dt, \quad 0 \le t \le T,$$

- Two risky assets:

$$\begin{aligned} \frac{dY(t)}{Y(t)} &= \mu_Y dt + \sigma_Y dW_Y(t), \quad 0 \le t \le T, \\ \frac{dF(t)}{F(t)} &= \mu_F dt + \sigma_F dW_F(t), \quad 0 \le t \le T, \end{aligned}$$

where the processes (W_Y, W_F) are correlated Brownian motions defined by

$$W_Y(t) = W_1(t), \quad W_F(t) = \rho W_1(t) + \sqrt{1-\rho^2} W_2(t),$$

and (W_1, W_2) are independent Brownian motions.

The insurance portfolio

- The lifetimes of the policyholders $(\tau_k)_{k=1,\dots,n}$ are independent and identically distributed:

$$\mathbb{P}\big(\tau_k > t\big) = e^{-\int_0^t \lambda(s)ds}, \quad 0 \le t \le T,$$

- The counting process and the compensated counting process:

$$N(t) = \sum_{k=1}^{n} \mathbf{1}\{\tau_k \le t\}, \quad \tilde{N}(t) = N(t) - \int_0^t (n - N(s-))\lambda(s)ds,$$

- The number of in-force policies:

$$J(t) = n - N(t),$$

- The benefit stream process:

$$\begin{aligned} B(t) &= \int_0^t (n - N(u-))A(u, Y(u), F(u))du + \int_0^t D(u, Y(u), F(u))dN(u) \\ &+ (n - N(T))S(Y(T), F(T))\mathbf{1}_{t=T}, \quad 0 \le t \le T. \end{aligned}$$

Hedgeable and non-hedgeable sources of risks

- Tradeable financial risk Y:

$$B(t) = S(Y(T))\mathbf{1}\{t = T\},$$

- Non-tradeable financial risk F:

$$B(t) = S(F(T))\mathbf{1}\{t = T\},$$

- Non-tradeable insurance risk N:

$$\begin{aligned} B(t) &= \int_0^t (n - N(u-))A(u)du + \int_0^t D(u)dN(u) \\ &\quad + (n - N(T))S\mathbf{1}\{t = T\}, \quad 0 \le t \le T, \end{aligned}$$

- Tradeable financial risk Y and non-tradeable insurance risk N:

$$\begin{aligned} B(t) &= \int_0^t (n - N(u-))A(u, Y(u))du + \int_0^t D(u, Y(u))dN(u) \\ &\quad + (n - N(T))S(Y(T))\mathbf{1}\{t = T\}, \quad 0 \le t \le T. \end{aligned}$$

Hedgeable and non-hedgeable sources of risks

- Tradeable financial risk Y: The fluctuations of the risky asset Y impact the payment process. This risk can be perfectly hedged by trading in Y,
- Non-tradeable financial risk F: The variations of the risky asset F impact the benefit stream as well. This risk can be partially hedged by trading in Y, since Y and F are correlated,
- Non-tradeable insurance risk N: The risk arises since the policyholders die at random times and the death-related benefits have to be paid at unpredictable times. This risk cannot be hedged since it is assumed to be independent of the financial market,
- How should be hedge and price the hedgeable and non-hedgeable risks?
- Our goal is to attach a *fair* value at any time t in $[0, T]$ to the future claims from the benefit stream B and to find the dynamic hedging strategy which underlies the *fair* value.

Arbitrage-free valuation

- The arbitrage-free value of the future claims from the process B:

$$\varphi_{B(t,T]}(t) = \mathbb{E}^{\mathbb{Q}}\Big[\int_t^T e^{-r(s-t)}dB(s)|\mathcal{F}_t\Big], \quad 0 \le t \le T,$$

where

$$\begin{aligned} \frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t &= \mathcal{M}^{\zeta,\chi}(t), \quad 0 \le t \le T, \\ \frac{d\mathcal{M}^{\zeta,\chi}(t)}{\mathcal{M}^{\zeta,\chi}(t-)} &= -\Big(\frac{\mu_Y - r}{\sigma_Y}\Big)dW_1(t) + \zeta(t)dW_2(t) + \chi(t)d\tilde{N}(t), \end{aligned}$$

- The processes (ζ, χ) are called risk premiums used for pricing the non-hedgeable financial and insurance risks,
- By $\hat{\mathbb{Q}}$ we denote the unique equivalent martingale measure in the complete financial market consisting of (R, Y), or the equivalent martingale measure in the combined financial and insurance model with zero risk premiums for the non-hedgeable risks.

Fair valuation

- The hedgeable benefit stream:

$$B^Y(t) = \int_0^t A(u, Y(u))du + S(Y(T))\mathbf{1}\{t = T\}, \quad 0 \le t \le T,$$

- The orthogonal benefit stream:

$$\begin{aligned} B^N(t) &= \int_0^t (n - N(u-))A(u)du + \int_0^t D(u)dN(u) \\ &\quad + (n - N(T))S\mathbf{1}\{t = T\}, \quad 0 \le t \le T, \end{aligned}$$

- We use the notions of *market-consistent, actuarial, fair valuation and hedging strategy* introduced by Jan Dhaene and Karim Barigou.

Fair valuation

Definition

Let $\varphi_{B(t,T]}(t)$ *denote the value at time* $t \in [0,T]$ *of the future claims from the process* B. *We will say that*

- *The valuation operator* φ *is market-consistent if for any process* B *and any hedgeable process* B^Y *we have that*

$$\varphi_{B(t,T]+B^Y(t,T]}(t) = \varphi_{B(t,T]}(t) + \varphi_{B^Y(t,T]}(t),$$

with

$$\varphi_{B^Y(t,T]}(t) = \mathbb{E}^{\hat{\mathbb{Q}}}\Big[\int_t^T e^{-r(u-t)} dB^Y(u) | \mathcal{F}_t^{W_1}\Big], \ 0 \le t \le T,$$

  where $\hat{\mathbb{Q}}$ *denotes the unique equivalent martingale measure for the traded risky asset* Y,

- *The valuation operator* φ *is actuarial if for any orthogonal process* B^N *we have that*

$$\varphi_{B^N(t,T]}(t) = \mathbb{E}^{\mathbb{P}}\Big[\int_t^T e^{-r(u-t)} dB^N(u) | \mathcal{F}_t^N\Big] + RM^{act}_{B^N(t,T]}(t), \quad 0 \le t \le T,$$

  where, for each $t \in [0,T]$, *the operator* $RM^{act}_{B^N(t,T]}(t)$ *is independent of the risky assets* (Y, F),

- *The valuation operator* φ *is fair if it is market-consistent and actuarial.*

Fair valuation

Definition

Let $(\theta_{B(t,T]}(s), t \leq s \leq T)$ denote the hedging strategy in the risky asset Y for the future claims from the process B. We will say that

- *The strategy θ is market-consistent if for any process B and any hedgeable process B^Y we have that*

$$\theta_{B(t,T]+B^Y(t,T]}(s) = \theta_{B(t,T]}(s) + v_y(s, Y(s))Y(s), \quad t \leq s \leq T,$$

  where $v(t,y) = \mathbb{E}^{\hat{\mathbb{Q}}}[\int_t^T e^{-r(u-t)} dB^Y(u)|Y(t) = y]$ and $\hat{\mathbb{Q}}$ denotes the unique equivalent martingale measure for the traded risky asset Y,

- *The strategy θ is actuarial if for any orthogonal process B^N we have that*

$$\theta_{B^N(t,T]}(s) = 0, \quad t \leq s \leq T,$$

- *The strategy θ is fair if it is market-consistent and actuarial.*

One-period valuation operator

- The idea is to split the valuation of the benefit stream into two parts:
- The first part should give a price of a hedgeable part of the benefit stream and should be related to the market cost of a hedging portfolio,
- The second part should give a price of a non-hedgeable part of the benefit stream left after the application of the hedging portfolio and should be related to the real-world cost of the claims,
- We have to make two decisions: How to define the hedging portfolio? And how to measure the risk of the remaining non-hedgeable claims?

One-period valuation operator - the hedging portfolio

- Let $\theta = (\theta(t), 0 \le t \le T)$ denote the dynamic hedging strategy – the amount of money invested in the risky asset Y,
- Let $V^\theta = (V^\theta(t), 0 \le t \le T)$ denote the self-financing hedging portfolio under the strategy θ given by the dynamics

$$\begin{aligned} dV^\theta(t) &= \theta(t)(\mu_Y dt + \sigma_Y dW_Y(t)) + (V^\theta(t) - \theta(t))rdt \\ &\quad -(n - N(t-))A(t, Y(t), F(t))dt - D(t, Y(t), F(t))dN(t), \end{aligned}$$

  and the terminal claims $(n - N(T))S(Y(T), F(T))$ are subtracted from $V^\theta(T)$ at time T,
- We minimize the mean-square hedging error at the terminal time under the equivalent martingale measure $\hat{\mathbb{Q}}$:

$$\inf_\theta \mathbb{E}^{\hat{\mathbb{Q}}}\left[\left|(n - N(T))S(Y(T), F(T)) - V^\theta(T)\right|^2\right].$$

One-period valuation operator - the hedging portfolio

Proposition

Let us define the functions:

$$v^k(t,y,f) = \mathbb{E}^{\hat{\mathbb{Q}}}_{t,y,f,k}\Big[\int_t^T e^{-r(u-t)}dB(u)\Big],$$
$$(t,y,f) \in [0,T] \times (0,\infty) \times (0,\infty),\ k \in \{0,\dots,n\}.$$

(i) The initial value of the hedging portfolio is given by

$$V_B^*(0) = v^n(0, Y(0), F(0)),$$

and the optimal dynamic hedging strategy is given by

$$\theta_B^*(t) = v_y^{J(t-)}(t, Y(t), F(t))Y(t) + v_f^{J(t-)}(t, Y(t), F(t))F(t)\frac{\sigma_F}{\sigma_Y}\rho, \quad 0 \le t \le T.$$

(ii) The optimal dynamic hedging strategy is market-consistent and actuarial, hence it is fair.

Remark: If we choose the real-world measure in the quadratic hedging objective, then the optimal strategy is not actuarial!

One-period valuation operator - the actuarial risk margin

- We introduce the one-period valuation operator:

$$\varrho(B) = V_B^*(0) + \pi\Big[\Big((n - N(T))S(Y(T), F(T)) - V_B^*(T)\Big)e^{-rT}\Big],$$

- The one-period actuarial valuation operator π can be decomposed into

$$\pi[\xi] = \mathbb{E}^{\mathbb{P}}[\xi] + RM[\xi],$$

where RM denotes a one-period actuarial risk margin operator,

- Consequently, we consider the one-period valuation operator:

$$\begin{aligned}\varrho(B) &= V_B^*(0) + \mathbb{E}^{\mathbb{P}}\Big[\Big((n - N(T))S(Y(T), F(T)) - V_B^*(T)\Big)e^{-rT}\Big] \\ &\quad + RM\Big[\Big((n - N(T))S(Y(T), F(T)) - V_B^*(T)\Big)e^{-rT}\Big] \\ &= \textit{Best Estimate of } B + \textit{Risk Margin for } B.\end{aligned}$$

One-period valuation operator

Proposition

Let us assume that the one-period actuarial risk margin RM satisfies the conditions of normalization and translation-invariance:

$$RM[0] = 0, \quad RM[\xi + a] = RM[\xi],$$

for any random variable ξ and constant a.
The one-period valuation operator is market-consistent and actuarial, hence it is fair.

Remarks:

- We do not include any assumptions on the risk premiums for the non-hedgeable risks when we solve our hedging problems,
- The risk premiums used for pricing the non-hedgeable risks should be implied by the subjective one-period actuarial valuation operator,
- We can disentangle hedgeable and non-hedgeable parts of the benefit stream and price them separately,
- Key examples of the one-period actuarial risk margin are standard deviation and variance,
- Antoon Pelsser and Ahmad Ghalehjooghi consider a different operator splitting for the one-period market-consistent and actuarial valuation operator.

Multi-period valuation operator

- Let us consider the time points $\mathcal{T} = 0, h, ..., T-h, T$.
- We iteratively and backward apply the one-period valuation operator with time step h:

$$\begin{aligned} \varphi_B(T) &= (n - N(T))S(Y(T), F(T)), \\ \varphi_B(t) &= \varrho_t\Big(\int_t^{t+h} d\tilde{B}(s)\Big), \quad t = 0, h, ..., T-h, \\ \tilde{B}(s) &= \int_t^s (n - N(u-))A(u, Y(u), F(u))du \\ &+ \int_t^s D(u, Y(u), F(u))dN(u) \\ &+ \varphi_B(t+h)\mathbf{1}\{s = t+h\}, \quad t \le s \le t+h, \end{aligned}$$

- We introduce the multi-period valuation operator:

$$\varphi_B(t) = V^*_{\tilde{B}}(t) + \mathbb{E}^{\mathbb{P}}\Big[\Big(\varphi_B(t+h) - V^*_{\tilde{B}}(t+h)\Big)e^{-rh}|\mathcal{F}_t\Big]$$
$$+RM\Big[\Big(\varphi_B(t+h) - V^*_{\tilde{B}}(t+h)\Big)e^{-rh}|\mathcal{F}_t\Big], \quad t = 0, h, ..., T-h.$$

Continuous-time valuation operator

- We would like to extend the definition of the price $\varphi_B(t)$ from $t \in \mathcal{T}$ to all times $t \in [0, T]$,
- The continuous-time valuation operator φ_B is defined as an operator which satisfies the continuous-time limit of the discrete-time pricing equation,
- We are interested in finding φ which satisfies

$$\lim_{h \to 0} \left\{ \frac{\mathbb{E}_{t,y,f,k}\Big[\big(\varphi(t+h) - V^*_{\tilde{B}}(t+h)\big)e^{-rh} - \big(\varphi(t+h) - V^*_{\tilde{B}}(t+h)\big)\Big]}{h} \right.$$
$$\left. + \frac{RM_{t,y,f,k}\Big[\big(\varphi(t+h) - V^*_{\tilde{B}}(t+h)\big)e^{-rh} - \big(\varphi(t+h) - V^*_{\tilde{B}}(t+h)\big)\Big]}{h} \right\} = 0,$$

for any $(t, y, f, k) \in [0, T) \times (0, \infty) \times (0, \infty) \times \{0, ..., n\}$.

Continuous-time valuation operator

Theorem

Let us consider the system of PDEs:

$$\begin{aligned}
&\varphi^k_t(t,y,f) + \varphi^k_y(t,y,f)yr + \varphi^k_f(t,y,f)f\big(\mu_F - \frac{\mu_Y - r}{\sigma_Y}\sigma_F\rho\big)\\
&+\varphi^k_{yf}(t,y,f)yf\sigma_Y\sigma_F\rho + \frac{1}{2}\varphi^k_{yy}(t,y,f)y^2\sigma_Y^2 + \frac{1}{2}\varphi^k_{ff}(t,y,f)f^2\sigma_F^2\\
&+kA(t,y,f) + \big(\varphi^{k-1}(t,y,f) + D(t,y,f) - \varphi^k(t,y,f)\big)k\lambda(t) - \varphi^k(t,y,f)r\\
&+\Phi^k\Big(t, \varphi^k_f(t,y,f)f\sigma_F\sqrt{1-\rho^2}, \varphi^{k-1}(t,y,f) + D(t,y,f) - \varphi^k(t,y,f)\Big) = 0,\\
&\qquad (t,y,f) \in [0,T) \times (0,\infty) \times (0,\infty),\\
&\varphi^k(T,y,f) = kS(y,f), \quad (y,f) \in (0,\infty) \times (0,\infty), \qquad (1)
\end{aligned}$$

for $k \in \{0,\ldots,n\}$, *where* $\Phi^k(t,x_1,x_2) = \frac{1}{2}\gamma(x_1^2 + x_2^2 k\lambda(t))$ *for the variance risk margin and* $\Phi^k(t,x_1,x_2) = \frac{1}{2}\gamma\sqrt{x_1^2 + x_2^2 k\lambda(t)}$ *for the standard deviation risk margin.*
We assume that there exist unique solutions $(\varphi^k)_{k=0,\ldots,n}$ *to the PDEs* (1).

Continuous-time valuation operator

Theorem

(i) The continuous-time valuation operator φ *determined by* (1) *satisfies the continuous-time limit of the discrete-time pricing equation as* $h \to 0$. *In particular, we prove*

$$\lim_{h\to 0}\frac{RM_{t,y,f,k}\Big[\big(\varphi(t+h)-V^*_{\tilde{B}}(t+h)\big)e^{-rh}-\big(\varphi(t+h)-V^*_{\tilde{B}}(t+h)\big)\Big]}{h}$$
$$=\Phi^k\Big(t,\varphi^k_f(t,y,f)f\sigma_F\sqrt{1-\rho^2},\varphi^{k-1}(t,y,f)+D(t,y,f)-\varphi^k(t,y,f)\Big).$$

(ii) The continuous-time valuation operator φ *is market-consistent and actuarial, hence it is fair.*
(iii) The hedging strategy which underlies the continuous-time valuation operator φ *is given by*

$$\begin{aligned}\vartheta^*(t) &= \varphi_y^{J(t-)}(t,Y(t),F(t))Y(t)\\ &\quad+\varphi_f^{J(t-)}(t,Y(t),F(t))F(t)\frac{\sigma_F}{\sigma_Y}\rho,\quad 0\le t\le T.\end{aligned}$$

The hedging strategy is market-consistent and actuarial, hence it is fair.

Remark: We can call Φ an instantaneous actuarial risk margin.

Continuous-time valuation operator

- The insurer's net asset value process:

$$NAV(t) = V_B^*(t) - \varphi(t), \quad 0 \le t \le T,$$

has the dynamics

$$\begin{aligned} dNAV(s) &= NAV(s)rds + \Phi(s)ds \\ &\quad -\varphi_f^{J(s-)}(s, Y(s), F(s))F(s)\sigma_F\sqrt{1-\rho^2}dW_2(s) \\ &\quad -\Big(\varphi^{J(s-)-1}(s, Y(s), F(s)) + D(s, Y(s), F(s)) - \varphi^{J(s-)}(s, Y(s), F(s))\Big)d\tilde{N}(s), \end{aligned}$$

- The first integral models the non-hedgeable risk that the value of the claims changes due to a change in the independent component of the risky asset F. The integrand is the delta-hedging perfect replication strategy for the independent component of the risky asset F,
- The second integral models the non-hedgeable risk that in the case of an independent event of the policyholder's death the insurer pays the death benefit and revaluates the claims for the in-force policies. The integrand is the sum at risk in the event of the policyholder's death.

Continuous-time valuation operator

Theorem

The continuous-time valuation operator has the representation:

$$\varphi^k(t,y,f) \quad = \quad \mathbb{E}^{\hat{\mathbb{Q}}}_{t,y,f,k}\Big[\int_t^T e^{-r(s-t)}dB(s) + \int_t^T e^{-r(s-t)}\Phi(s)ds\Big],$$
$$(t,y,f) \in [0,T] \times (0,\infty) \times (0,\infty),\ k \in \{0,...,n\}.$$

- The valuation operator values liabilities as the best estimate of the liability plus the total actuarial risk margin for the liability:

$$\begin{aligned} \varphi \quad &= \quad Fair\ Value\ of\ B \\ &= \quad Best\ Estimate\ of\ B\ +\ Total\ Actuarial\ Risk\ Margin\ for\ B. \end{aligned}$$

Continuous-time valuation operator

The solvency position, the unwind of the risk margin and the profit recognition over the duration of the portfolio:

- At each time $t \in [0, T)$, the insurer must hold an additional capital determined by $\Phi(t)$ which protects the insurer against adverse scenarios in the evolution of the non-hedgeable risks,
- At time $t = 0$ the expected cost (the best estimate) of providing the additional capitals Φ till maturity of the insurance portfolio is equal to $\mathbb{E}^{\hat{\mathbb{Q}}}[\int_0^T e^{-rs}\Phi(s)ds]$,
- As the time passes, the technical provision (the price φ of the benefit stream), the best estimate of the liability and the cost of financing the future instantaneous actuarial risk margins are recalculated,
- The instantaneous actuarial risk margins are released from the technical provision as the time passes and, on average, they are not used to cover the losses since the realized loss on the hedgeable risk is always zero and the expected loss on the non-hedgeable risks is also zero, both under $\mathbb{P}$ and $\hat{\mathbb{Q}}$,
- The insurer earns, on average, a risk-free rate on the net asset value and the instantaneous actuarial risk margins accumulated with the risk-free rate:

$$\mathbb{E}[NAV(t+h)e^{-rh}|\mathcal{F}_t] = NAV(t) + \mathbb{E}\Big[\int_t^{t+h} e^{-ru}\Phi(u)du|\mathcal{F}_t\Big].$$

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Thank you very much.

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