VALUTAZIONE DELLA RISERVA SINISTRI:
DAL METODO DELLA CATENA AGLI HGLM

Liviana Picech

Università di Trieste

VALUTAZIONE DELLA RISERVA SINISTRI:
DAL METODO DELLA CATENA AGLI HGLM

Gigante P., Picech L., Sigalotti L.,

Claims reserving in the hierarchical generalized linear model framework, *Insurance: Mathematics and Economics* 52, 2013

Prediction error for credible claims reverves: an *h*-likelihood approach, Eur. Actuar. J. 3(2), 2013

A mixture model for payments and payment numbers in claims reserving, ASTIN Bulletin 48(1), 2016

Calendar year effect modeling for claims reserving in HGLM, ASTIN Bulletin 49(3), 2019

VALUTAZIONE DELLA RISERVA SINISTRI:

DAL METODO DELLA CATENA AGLI HGLM

Overview

- Introduction

- Chain-ladder model

- Conjugate HGLM, HGLM and quasi-HGLM

- Bornhuetter-Ferguson model

- Overdispersed Poisson-gamma model

- HGLM estimates for credible claims reserves

- A Poisson-gamma HGLM with diagonal effects

INTRODUCTION

t the maximum development year

available information at the end of year t

$$y_{ij}: \quad i,j = 0,\ldots,t, \quad i+j \le t$$

➢ probabilistic assumptions for the

incremental payments

$$\left\{ Y_{ij} \mid i,j = 0,\ldots,t \right\}$$

Origin years	Development years							
	0	1		j				t
0								
1								
i	Y_{i0}			Y_{ij}	Y_{it-i}			Y_{it}
					unknown			
t								

run-off table

➢ **Predictions** of future payments: $Y_{ij}: \quad i+j > t$; **prediction error** evaluations

- Claims reserve for origin year i $\quad R_i = \sum\limits_{j=t-i+1}^{t} Y_{ij}, \quad i = 1,\ldots,t$

- Total claims reserve $\quad R = \sum\limits_{i=1}^{t} R_i = \sum\limits_{i,j:i+j>t} Y_{ij}$

4

CHAIN-LADDER MODEL

CL algorithm

observations $\quad c_{ij} = \sum_{k=0}^{j} y_{ik}$ $\quad$ **cumulative claims**

CL estimates for the **link ratios** (development factors):

$$\hat{f}_j = \frac{\sum\limits_{i=0}^{t-j-1} c_{i,j+1}}{\sum\limits_{i=0}^{t-j-1} c_{ij}} \qquad j = 0,...,t-1$$

	0		j	$j+1$			t
0							
1							
$t-j-1$							
t							

CL estimate for the ultimate claims C_{it} : $\qquad \hat{C}_{it}^{CL} = c_{i,t-i} \prod\limits_{j=t-i}^{t-1} \hat{f}_j$

CL estimates for the claims reserve: $\qquad \hat{R}_i^{CL} = \hat{C}_{it}^{CL} - c_{i,t-i}$

Poisson model

Y_{ij} independent, Poisson distributed with

$$E(Y_{ij}) = a_i \gamma_j \qquad \text{for all } i = 0,\ldots,t, \; j = 0,\ldots,t \qquad \text{and } \sum_{j=0}^{t} \gamma_j = 1$$

➢ $E(C_{it}) = a_i \qquad i = 0,\ldots,t$ expected value of the ultimate claims for origin year i

➢ $\gamma_j \qquad\qquad j = 0,\ldots,t$ expected "cash-flow pattern" (rate of the ultimate claims paid in development period j)

Let the **maximum likelihood (ML) estimates** of the parameters of the Poisson model

$$\hat{a}_i^{POI}, \qquad i = 0,\ldots,t, \qquad\qquad \hat{\gamma}_j^{POI}, \qquad j = 0,\ldots,t$$

Poisson ML estimate for the **ultimate claims**: $\hat{C}_{it}^{POI} = c_{i,t-i} + \hat{a}_i^{POI} \sum_{j=t-i+1}^{t} \hat{\gamma}_j^{POI}$

Poisson ML estimate for the **claims reserve**: $\hat{R}_{it}^{POI} = \hat{a}_i^{POI} \sum_{j=t-i+1}^{t} \hat{\gamma}_j^{POI}$

Remark: $\hat{C}_{it}^{POI} = \hat{C}_{it}^{CL} \qquad\qquad \hat{R}_{it}^{POI} = \hat{R}_{it}^{CL}$

Remark:

➢ Poisson model:

- the incremental payments Y_{ij} are independent Poisson distributed

- $E(Y_{ij}) = a_i \gamma_j \Leftrightarrow \log(E(Y_{ij})) = \log(a_i) + \log(\gamma_j)$ for all $i = 0,\ldots,t$, $j = 0,\ldots,t$

➢ **Generalized Linear Model** (quasi-GLM, Overdispersed Poisson)

a1) Independence assumption

Y_{ij} independent $i = 0,\ldots,t$, $j = 0,\ldots,t$

a2) Distributional assumption

$Y_{ij} \sim EDF$ $E(Y_{ij}) = \mu_{ij}$, $var(Y_{ij}) = \phi V(\mu_{ij})$

a3) Structural assumption

$g(\mu_{ij}) = x_{ij}^T \boldsymbol{\beta}$

CONJUGATE HGLM, HGLM AND QUASI-HGLM

Lee, Nelder, Pawitan (2006),   Generalized linear models with random effects: a unified
approach via h-likelihood

A claims reserving model in the HGLM framework

Let

Y_{ij} response variables, incremental payments $i, j = 0,1,...,t$

U_i **unobservable risk parameters** (origin years) $i = 0,...,t$

$\boldsymbol{x}_{ij}$ vector of covariates

Model assumptions (Conjugate HGLM)

a1) *Independence assumptions*

$U_0, \dots, U_t$ independent

$Y_{ij}|U = u$, $i = 0,\dots,t$, $j = 0,\dots,t$, independent for any value u of $U = (U_0,\dots,U_t)$

$$\left(Y_{ij}|U = u\right) \overset{d}{=} \left(Y_{ij}|U_i = u_i\right)$$

a2) *Distributional assumptions for the responses conditional on risk parameters*

$$\left(Y_{ij}|U_i = u_i\right) \sim EDF$$

$$E(Y_{ij}|U_i = u_i) = b'(\theta_{ij}) = \mu_{ij}, \qquad var(Y_{ij}|U_i = u_i) = \frac{\phi}{\omega_{ij}} V(\mu_{ij})$$

with ω_{ij}, $i, j = 0,\dots,t$, known weights (e.g. exposure measure)

a3) *Structural assumptions*

$$\mu_{ij} = g^{-1}(x_{ij}^T \beta + w_i) \qquad \text{where} \quad g = b'^{-1} \quad \text{(\underline{canonical link})}$$

β **fixed effects parameters**

$w_i = g(u_i)$ **random effects**

a4) *Distributional assumptions for the risk parameters*

$W_i = g(U_i) = b'^{-1}(U_i)$ distribution conjugate of the EDF of $Y_{ij}|U_i = u_i$,

$$f_{W_i}(w) = exp\left\{ \frac{1}{\lambda_i} \left[\psi_i w - b(w) \right] \right\} d(\psi_i, \lambda_i)$$

where ψ_i and λ_i are **hyperparameters** (λ_i is the **dispersion parameter**)

➢ $\psi_i = E(U_i)$

➢ $\lambda_i = \mathrm{var}(U_i) / E(U_i^p)$ in Tweedie models (where $V(\mu) = \mu^p$ for some *p*)

(Jewell (1974), Bühlmann, Gisler (2005), Ohlsson, Johansson (2006))

Parameter estimation:

Hierarchical log-likelihood or ***h-likekelihood*** (Lee, Nelder (1996))

$$h \equiv log\, f_{Y,W} = l_{Y|W=w} + l_W$$

where $l_{Y|W=w}$ is the log-likelihood of $Y|W=w$

        l_W is the logarithm of the density of W

Given the data y and the hyperparameters ψ, if the dispersion parameters ϕ and λ are known, the maximum h-likelihood estimates of the fixed effect parameters and the random effects are the solution of

$$\begin{cases} \partial h / \partial \boldsymbol{\beta} = \mathbf{0} \\ \partial h / \partial \boldsymbol{w} = \mathbf{0} \end{cases}$$

with $h(\boldsymbol{\beta}, \boldsymbol{w}; \phi, \lambda; \boldsymbol{y}, \boldsymbol{\psi}, \boldsymbol{\omega}) = \sum\limits_{\substack{i,j \\ i+j \le t}} \frac{\omega_{ij}}{\phi} \left[y_{ij}(\boldsymbol{x}_{ij}^T \boldsymbol{\beta} + w_i) - b(\boldsymbol{x}_{ij}^T \boldsymbol{\beta} + w_i) \right] + \sum\limits_{i=0}^{t} \frac{\omega_i}{\phi} \left[\psi_i w_i - b(w_i) \right]$

where $\theta_{ij} = \boldsymbol{x}_{ij}^T \boldsymbol{\beta} + w_i$ because of the canonical link; $\omega_i = \phi / \lambda_i$.

The *h*-likelihood $h(\boldsymbol{\beta}, \boldsymbol{w}; \phi, \lambda; \boldsymbol{y}, \boldsymbol{\psi}, \boldsymbol{\omega})$ can be viewed as the log-likelihood of an ***augmented GLM***.

11

HGLM AND QUASI-HGLM

Conjugate HGLM can be extended.

HGLM

- The link function g is not necessarily the canonical one

- transform of the risk parameter in the linear predictor $w_i = g_1(u_i)$; not necessarily $g = g_1$

- $W_i = g_1(U_i)$, follows a distribution conjugate of an EDF with cumulant b_1 (or variance function V_1); not necessarily $b_1 = b$ (or $V_1 = V$).

Quasi-HGLM

Provides an extension similar to that leading from GLMs to quasi-GLMs:

- no full distributional assumptions are necessary

- the dispersion parameters ϕ e λ_i can be estimated through a regression structure:

$$\phi_{ij} = g_\phi^{-1}(\boldsymbol{x}_{\phi,ij}^T \boldsymbol{\gamma}_\phi), \qquad \lambda_i = g_\lambda^{-1}(\boldsymbol{x}_{\lambda,i}^T \boldsymbol{\gamma}_\lambda).$$

A quasi-HGLM can be fitted by **estimating iteratively three interconnected GLMs or quasi-GLMs**.

12

BORNHUETTER-FERGUSON MODEL

Let

$$E(C_{it}) = \mu_i \qquad i = 0,\ldots,t \qquad \text{expected value of the ultimate claims}$$

$$0 < b_0 \leq b_1 \leq \cdots \leq b_t = 1 \qquad \text{expected claims development pattern}$$

$$b_j \quad \text{rate of the ultimate claims expected to be paid within development year j}$$

BF estimate for the **ultimate claims** C_{it} : $\qquad \hat{C}_{it}^{BF} = c_{i,t-i} + \hat{\mu}_i(1 - \hat{b}_{t-i})$

BF estimates for the **claims reserve**: $\qquad \hat{R}_{it}^{BF} = \hat{\mu}_i(1 - \hat{b}_{t-i})$

where $\qquad \hat{\mu}_i \qquad \qquad \qquad$ "initial" estimates of the ultimate claims

$$\hat{b}_{t-i} = \prod_{j=t-i}^{t-1} \hat{\lambda}_j^{-1} \qquad \text{CL estimate of the development pattern}$$

In fact, $\quad \hat{C}_{it}^{CL} = c_{i,t-i} \prod_{j=t-i}^{t-1} \hat{f}_j \qquad c_{i,t-i} = \hat{C}_{it}^{CL}\left(\prod_{j=t-i}^{t-1} \hat{\lambda}_j\right)^{-1} = \hat{C}_{it}^{CL}\ \hat{b}_{t-i}$

$$\hat{R}_{it}^{CL} = \hat{C}_{it}^{CL} - \hat{C}_{it}^{CL}\ \hat{b}_{t-i} = \hat{C}_{it}^{CL}(1 - \hat{b}_{t-i}) = \frac{c_{i,t-i}}{\hat{b}_{t-i}}(1 - \hat{b}_{t-i})$$

The two "extreme position" can be combined $\implies$ **CREDIBLE CLAIMS RESERVES**

OVERDISPERSED POISSON-GAMMA MODEL

Let

$$\left\{ Y_{ij} \mid i, j = 0,\ldots,t \right\}$$ hierarchical overdispersed Poisson (ODP) - gamma

(England, Verrall (2002); Verrall (2004); Wüthrich (2007))

with

- Y_{ij}, $i, j = 0,\ldots,t$, *response variables* (incremental payments)

- $U = (U_0,\ldots,U_t)$ **unobservable risk parameters** related to the origin years,

1) *Independence assumptions*

$U_0, \ldots, U_t$ are independent,

$Y_{ij} \mid U = u$, $i, j = 0,\ldots,t$, are independent for any value u of U,

$$(Y_{i0},\ldots,Y_{it}) \mid U = u \overset{d}{=} (Y_{i0},\ldots,Y_{it}) \mid U_i = u_i$$

2) ***Distributional assumptions for the responses conditional on risk parameters***

$$Y_{ij}|U_i = u_i \sim \text{ODP}, \qquad E\left(Y_{ij}|U_i = u_i\right) = \mu_{ij}, \qquad Var(Y_{ij}|U_i = u_i) = \phi\mu_{ij}$$

3) ***Structural assumption***

$$\mu_{ij} = exp\left(log(u_i) + \beta_j\right) = u_i exp(\beta_j)$$

4) ***Distributional assumptions for the risk parameters***

$$U_i \sim \text{Gamma}, \qquad E(U_i) = \psi_i, \qquad var(U_i)/E(U_i) = \lambda_i$$

where ψ_i and λ_i ***hyperparameters***; λ_i ***dispersion parameter***

➤ $E(Y_{ij}|U_i = u_i) = u_i \exp(\beta_j) = \exp(\beta_j + w_i)$, with $w_i = \log(u_i)$

➤ $W_i = \log(U_i)$ follows a distribution conjugate of the ODP of $Y_{ij}|U_i = u_i$,

➤ The assumptions 1), 2), 3), 4) define a quasi-**HGLM**

with β_j, **fixed effects**, w_i, **random effects**

Some well-known results (e.g. Bühlmann, Gisler (2005); Verrall (2004); Wüthrich (2007))

Given   the parameters $\beta_j, \; j = 0,\ldots,t,$ and ϕ

               the hyperparameters ψ_i and $\lambda_i,$ $i = 0,\ldots,t$

$$\mathcal{D}_t = \left\{ Y_{ij} : \quad i + j \leq t \right\}$$

$$\Rightarrow \quad E(U_i|\mathcal{D}_t) = z_i \, \frac{\sum\limits_{j=0}^{t-i} Y_{ij}}{\sum\limits_{j=0}^{t-i} \exp(\beta_j)} + (1 - z_i)\,\psi_i \qquad \text{where} \qquad z_i = \frac{\sum\limits_{j=0}^{t-i} \exp(\beta_j)}{\sum\limits_{j=0}^{t-i} \exp(\beta_j) + \dfrac{\phi}{\lambda_i}};$$

$$\Rightarrow \quad E(R_i|\mathcal{D}_t) = \left[z_i \, \frac{\sum\limits_{j=0}^{t-i} Y_{ij}}{\sum\limits_{j=0}^{t-i} \exp(\beta_j)} + (1 - z_i)\,\psi_i \right] \sum\limits_{j=t-i+1}^{t} \exp(\beta_j) \qquad \text{Bayesian estimator of } R_i$$

We have (e.g. England, Verrall (2002); Wüthrich (2007)):

$$E(R_i|\mathcal{D}_t) = \left[z_i \frac{C_{i,t-i}}{b_{t-i}} + (1 - z_i) \, E(C_{it}) \right](1 - b_{t-i})$$

with $b_j = \dfrac{\sum\limits_{h=0}^{j} \exp(\beta_h)}{\sum\limits_{h=0}^{t} \exp(\beta_h)}, \quad j = 0,\ldots,t$ the claims development pattern; $E(C_{it}) = \psi_i \sum\limits_{h=0}^{t} \exp(\beta_h)$

$$\Rightarrow \qquad E(R_i|\mathcal{D}_t) = z_i \, \tilde{R}_{it}^{BCL} + (1 - z_i) \, \tilde{R}_{it}^{BBF}$$

Mixture of CL and BF estimators:

	Bayesian estimators
CL model (Mack(1993))	$\tilde{R}_i^{BCL} = \dfrac{C_{i,t-i}}{b_{t-i}}(1 - b_{t-i})$
BF model (Mack (2008); Wüthrich, Merz (2008))	$\tilde{R}_i^{BBF} = \mu_i(1 - b_{t-i})$

if, in the ODP-Gamma, CL and BF models, the parameters are such that the claims development pattern b_j, $j = 0,\ldots,t$ is the same and $E(C_{it}) = \mu_i$.

HGLM ESTIMATES FOR CREDIBLE CLAIMS RESERVES

Given: the data of the run-off triangle $\qquad y_{ij}, \quad i, j = 0, \ldots, t, \quad i + j \leq t$

the "external data", initial estimate of ultimate claims $\quad \mu_0^{(i)}, \qquad i = 0, \ldots, t$

We set $\qquad \psi_i = E(U_i) = \mu_0^{(i)}, \qquad i = 0, \ldots, t$

We estimate

the fixed effect parameters $\qquad \beta_j, \ j = 0, \ldots, t$

the random effects $\qquad w_i = \log(u_i), \ i = 0, \ldots, t$

the dispersion parameters $\qquad \phi \quad$ and $\quad \lambda_i, \ i = 0, \ldots, t$

Given the estimates of the dispersion parameters, the quasi-HGLM estimates of the fixed and random effects satisfy the following conditions:

$$
\begin{cases}
\exp(\hat{\beta}_j) = \sum_{i=0}^{t-j} y_{ij} \Big/ \sum_{i=0}^{t-j} \hat{u}_i & j = 0,\ldots,t \\[2em]
\hat{u}_i = \hat{z}_i \dfrac{\sum_{j=0}^{t-i} y_{ij}}{\sum_{j=0}^{t-i} \exp(\hat{\beta}_j)} + (1-\hat{z}_i)\psi_i & i = 0,\ldots,t, \qquad with \quad \hat{z}_i = \dfrac{\sum_{j=0}^{t-i} \exp(\hat{\beta}_j)}{\sum_{j=0}^{t-i} \exp(\hat{\beta}_j) + \dfrac{\hat{\phi}}{\hat{\lambda}_i}}
\end{cases}
$$

Compare with

$$
E(U_i | \mathcal{D}_t) = z_i \dfrac{\sum_{j=0}^{t-i} Y_{ij}}{\sum_{j=0}^{t-i} \exp(\beta_j)} + (1-z_i)\psi_i
\qquad\qquad
z_i = \dfrac{\sum_{j=0}^{t-i} \exp(\beta_j)}{\sum_{j=0}^{t-i} \exp(\beta_j) + \dfrac{\phi}{\lambda_i}}
$$

Prediction

For the claims reserve $\qquad\qquad R = \sum_{i=1}^{t} R_i = \sum_{i,j:i+j>t} Y_{ij}$ $\qquad$ we have

$$\overline{R} = E(R|U) = \sum_{i,j:i+j>t} \exp(\beta_j + W_i)$$

and consider the estimator

$$\widetilde{R} = \sum_{i,j:i+j>t} \exp(\widetilde{\beta}_j + \widetilde{w}_i)$$

where $\widetilde{\beta}$ and $\widetilde{w}$ are the quasi-HGLM estimators of β and W.

HGLM estimates of the claims reserves

$$\hat{R}_i = \hat{u}_i \sum_{j=t-i+1}^{t} \exp(\hat{\beta}_j) = \left[\hat{z}_i \frac{\sum_{j=0}^{t-i} y_{ij}}{\sum_{j=0}^{t-i} \exp(\hat{\beta}_j)} + (1-\hat{z}_i)\psi_i \right] \sum_{j=t-i+1}^{t} \exp(\hat{\beta}_j)$$

$$= \left[\hat{z}_i \frac{\sum_{j=0}^{t-i} y_{ij}}{\hat{b}_{t-i}} + (1-\hat{z}_i)\psi_i \sum_{j=0}^{t} \exp(\hat{\beta}_j) \right] (1-\hat{b}_{t-i}) = \hat{z}_i \hat{R}_i^{BCL} + (1-\hat{z}_i)\hat{R}_i^{BBF}$$

where

- $\hat{b}_j = \sum_{h=0}^{j} \exp(\hat{\beta}_h) \Big/ \sum_{h=0}^{t} \exp(\hat{\beta}_h), \qquad j = 0,\ldots,t$

- $\hat{R}_i^{BCL} = \frac{\sum_{j=0}^{t-i} y_{ij}}{\hat{b}_{t-i}}(1-\hat{b}_{t-i}), \qquad \hat{R}_i^{BBF} = \left(\psi_i \sum_{j=0}^{t} \exp(\hat{\beta}_j) \right)(1-\hat{b}_{t-i})$

- $\psi_i \sum_{j=0}^{t} \exp(\hat{\beta}_j)$ is an estimate of $E(C_{it}) = \psi_i \sum_{h=0}^{t} \exp(\beta_h)$

21

Conditional mean square error of prediction

$$MSEP_{R|\mathcal{D}_t}(\tilde{R}) = E\left[\left(R - \tilde{R}\right)^2 \big| \mathcal{D}_t\right]$$

$$= E\left[Var(R|U)|\mathcal{D}_t\right] + var(\bar{R}|\mathcal{D}_t) + E\left[\left(E(\bar{R}|\mathcal{D}_t) - \tilde{R}\right)^2 \big| \mathcal{D}_t\right]$$

Following Booth, Hobert (1998) and Lee, Ha (2009), estimates of the three terms are:

- $\hat{E}\left[Var(R|U)|\mathcal{D}_t\right] = \sum\limits_{i,j:i+j>t} \hat{\phi} \exp(\hat{\beta}_j + \hat{w}_i)$

- $\widehat{var}(\bar{R}|\mathcal{D}_t) = \widehat{var}(r(W)|\mathcal{D}_t) \approx J_r(w) \, \boldsymbol{H}_{22}^{-1} J_r(w)^T \big|_{\hat{\delta}}$

- $\hat{E}\left[\left(E(\bar{R}|\mathcal{D}_t) - \tilde{R}\right)^2 \big| \mathcal{D}_t\right] \approx \hat{E}\left[\left(f(\boldsymbol{\beta}) - f(\tilde{\boldsymbol{\beta}})\right)^2 \big| \mathcal{D}_t\right] \approx J_f(\boldsymbol{\beta}) \, \boldsymbol{G}^{-1} J_f(\boldsymbol{\beta})^T \big|_{\hat{\delta}}$

where

- $J_r(w)$ and $J_f(\beta)$ the Jacobian matrices of the functions

$$r(w) = \sum_{i,j:i+j>t} \exp(\beta_j + w_i) \qquad f(\beta) = \sum_{i,j:i+j>t} g^{-1}(x_{ij}^T \beta + \tilde{w}_i(\beta))$$

with $\tilde{w}_i(\beta)$ the HGLM estimator of w_i for a given β

- $\mathbf{\mathcal{I}}(\delta) = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{12}^T & \mathbf{H}_{22} \end{bmatrix}, \qquad \mathbf{\mathcal{I}}(\delta)^{-1} = \begin{bmatrix} \mathbf{G}^{-1} & \mathbf{F} \\ \mathbf{F}^T & \mathbf{C} \end{bmatrix}$

Remark:

The quasi-HGLM is fitted by **estimating iteratively three interconnected quasi-GLMs** and the first one is an augmented GLM for the run-off data y and the pseudo-data ψ

At convergence, the inverse of the Fisher information matrix $\mathbf{\mathcal{I}}(\hat{\delta})^{-1}$, $\delta = (\beta^T, w^T)^T$, of this augmented GLM provides an estimate of the **variance-covariance matrix**

$$var\begin{pmatrix} \tilde{\beta} \\ \tilde{w} - W \end{pmatrix}$$

EXAMPLE

Data: Tables 2.4-2.5 in Wüthrich, Merz (2008) – Borhuetter-Ferguson method.

Example: ODP-Gamma, quasi-HGLM model with $\phi_{ij} = \phi$ and $\lambda_i = \lambda$.

Origin year i	Initial estimate ψ_i /1000	Quasi-HGLM		
		$\hat{u}_i$ /1000	$\displaystyle\sum_{j>t-i} \exp(\hat{\beta}_j)$ × 100	$\hat{R}_i$
1	11,367	11,906	0.1277	15,199
2	10,963	11,799	0.2214	26,125
3	10,617	10,952	0.3183	34,857
4	11,045	11,159	0.7762	86,623
5	11,481	11,459	1.3909	159,377
6	11,414	11,006	2.6763	294,565
7	11,127	10,219	4.6062	470,703
8	10,987	10,190	10.6646	1,086,682
9	11,618	11,194	36.2808	4,061,355

$$\hat{R}_i = \hat{z}_i \hat{R}_i^{BCL} + (1 - \hat{z}_i)\hat{R}_i^{BBF}$$

| Origin year i | $\hat{R}_i$ | Quasi-HGLM | | | | $\hat{R}_i^{CL}$ | $\hat{R}_i^{BF}$ | $1 - \hat{b}_{t-i}^{CL}$ |
		$\hat{z}_i$	$\hat{R}_i^{BCL}$	$\hat{R}_i^{BBF}$	$1 - \hat{b}_{t-i}$			
1	15,199	0.7391	15,442	14,511	0.00145	15,125	16,124	0.00142
2	26,125	0.7389	26,780	24,274	0.00251	26,257	26,998	0.00246
3	34,857	0.7387	35,234	33,791	0.00361	34,538	37,575	0.00354
4	86,623	0.7377	86,939	85,734	0.00880	85,301	95,434	0.00864
5	159,377	0.7363	159,268	159,682	0.01578	156,494	178,023	0.01551
6	294,565	0.7334	290,603	305,462	0.03036	286,121	341,306	0.02990
7	470,703	0.7289	455,156	512,508	0.05225	449,167	574,090	0.05160
8	1,086,682	0.7138	1,052,603	1,171,674	0.12097	1,043,243	1,318,646	0.12002
9	4,061,355	0.6254	3,969,176	4,215,257	0.41154	3,950,815	4,768,385	0.41042

Remark: $\quad \hat{R}_i^{BBF} = \left[\mu_0^{(i)} \sum_{j=0}^{t} \exp(\hat{\beta}_j) \right](1 - \hat{b}_{t-i}), \quad \hat{R}_i^{BF} = \mu_0^{(i)} (1 - \hat{b}_{t-i}^{CL}), \quad \sum_{j=0}^{t} \exp(\hat{\beta}_j) = 0.88159$

$$\hat{\phi} = 14,895 \qquad \hat{\lambda} = 47,936 \qquad \hat{\phi}/\hat{\lambda} = 0.31$$

Model	Reserve	RMSEP	RMSEP %
HGLM	6,235,486	419,505	0.067
CL-ODP	6,047,061	429,891	0.071
BF(Alai *et al.* (2009))	7,356,575	471,971	0.064
Bühlmann-Straub(Wüthrich,Merz(2008))	6,159,709	317,540	0.052
Bayesian(England *et al.* (2012)) - a=100	6,145,526	422,526	0.069
Bayesian(England *et al.* (2012)) - $a \to \infty$	6,664,047	395,012	0.059

	HGLM	CL-ODP
$\hat{\phi}$	14,895	14,714

A POISSON-GAMMA HGLM WITH DIAGONAL EFFECTS

Let

Y_{ij} response variables, incremental payments $i, j = 0,1,...,t$

$(\boldsymbol{U},\boldsymbol{V}) = (U_0,...,U_t,V_0,...,V_{2t})$ **unobservable risk parameters**

$U_i,$ $i = 0,...,t,$ is related to the origin year i

V_{i+j} $i, j = 0,...,t$ is related to the calendar year or payment year $i+j$

a1) *Independence assumptions*

The risk parameters $(\boldsymbol{U},\boldsymbol{V}) = (U_0,...,U_t,V_0,...,V_{2t})$ are independent,

Conditionally on $(\boldsymbol{U},\boldsymbol{V})$, Y_{ij}, $i, j = 0,...,t$, are independent,

$$\left[Y_{ij} | (\boldsymbol{U},\boldsymbol{V}) = (\boldsymbol{u},\boldsymbol{v})\right] \overset{d}{=} \left[Y_{ij} | (U_i,V_{i+j}) = (u_i,v_{i+j})\right], \ i, j = 0,...,t, \text{ for any } (\boldsymbol{u},\boldsymbol{v})$$

a2) *Distributional assumptions for the responses, conditional on the risk parameters*

$$\left[Y_{ij}\middle|\left(U_i,V_{i+j}\right)=\left(u_i,v_{i+j}\right)\right] \text{ are overdispersed Poisson distributed}$$

$$E\left[Y_{ij}\middle|\left(U_i,V_{i+j}\right)=\left(u_i,v_{i+j}\right)\right]=\mu_{ij} \qquad var\left[Y_{ij}\middle|\left(U_i,V_{i+j}\right)=\left(u_i,v_{i+j}\right)\right]=\phi_{ij}\,\mu_{ij}$$

a3) *Structural assumptions for the expectations*

$$E\left[Y_{ij}\middle|\left(U_i,V_{i+j}\right)=\left(u_i,v_{i+j}\right)\right]=\mu_{ij}=\exp(\beta_j+w_{U,i}+w_{V,i+j})=e^{\beta_j}u_iv_{i+j}$$

where β_j regression parameter (fixed effect) for development year j

$w_{U,i}=\log(u_i)$ and $w_{V,i+j}=\log(v_{i+j})$ are the random effects

a4) *Distributional assumptions for the risk parameters*

Let $\quad W_{U,i} = \log(U_i) \qquad\qquad W_{V,i+j} = \log(V_{i+j})$

with $U_i \sim$ Gamma, $\qquad E(U_i) = \psi_{U,i}, \qquad\qquad var(U_i)/E(U_i) = \lambda_{U,i}$

with $V_{i+j} \sim$ Gamma, $\qquad E(V_{i+j}) = \psi_{V,i+j}, \qquad var(V_{i+j})/E(V_{i+j}) = \lambda_{V,i+j}$

where $\quad \psi_{U,i}$, $\psi_{V,i+j}, \qquad \lambda_{U,i}$ and $\lambda_{V,i+j} \qquad$ *hyperparameters*;

$\qquad\qquad\qquad\qquad\qquad\qquad \lambda_{U,i}$ and $\lambda_{V,i+j} \qquad$ *dispersion parameter*

To estimate the dispersion parameters $\quad \phi_{ij}$, $\lambda_{U,i}$, $\lambda_{V,i+j}$, $i, j = 0,\ldots,t,$ we make the following assumptions

a5) *Structural assumptions for the dispersions*

$$\phi_{ij} = \phi, \qquad i, j = 0,\ldots,t,$$

$$\lambda_{U,i} = \lambda_U, \qquad i = 0,\ldots,t, \qquad\qquad \lambda_{V,i+j} = \lambda_V, \qquad i + j = 0, \ldots, 2t$$

We estimate a quasi-Hierarchical Generalized Linear Model

Remark:

The maximum h-likelihood estimates of the fixed effect parameters and the random effects are the solution of

$$\begin{cases} \partial h / \partial \boldsymbol{\beta} = \mathbf{0} \\ \partial h / \partial \boldsymbol{w} = \mathbf{0} \end{cases}$$

We get

$$\widehat{w}_{V,i+j} = log(\psi_{V,i+j}) \qquad \text{for} \quad i+j > t$$

Prediction

For the claims reserve $\qquad R = \sum_{i=1}^{t} R_i = \sum_{i,j:i+j>t} Y_{ij}$

we consider the estimator $\qquad \tilde{R} = \sum_{i,j:i+j>t} exp(\tilde{\beta}_j + \tilde{w}_{U,i} + \tilde{w}_{V,i+j})$

A Poisson-gamma HGLM with diagonal effects

HGLM estimate of the claims reserve

$$\hat{R}_i = \hat{u}_i \sum_{j=t-i+1}^{t} exp(\hat{\beta}_j)\, \hat{v}_{i+j}$$

$$= \left(\hat{z}_{U,i} \frac{\sum_{j=0}^{t-i} y_{ij}\, \hat{v}_{i+j}}{\sum_{j=0}^{t-i} exp(\hat{\beta}_j)\, \hat{v}_{i+j}} + (1 - \hat{z}_{U,i})\psi_{U,i} \right) \sum_{j=t-i+1}^{t} exp(\hat{\beta}_j)\, \hat{v}_{i+j}$$

Conditional mean square error of prediction

$$MSEP_{R|\mathcal{D}_t}(\tilde{R}) = E\left[(R - \tilde{R})^2 \big| \mathcal{D}_t \right] = E\left[var(R|U,V)|\mathcal{D}_t \right] + var(E(R|U,V)|\mathcal{D}_t) + \left(E(R|\mathcal{D}_t) - \tilde{R} \right)^2$$

can be estimated in a similar way as before.

EXAMPLE

Data: Tables 2.4-2.5 in Wüthrich, Merz (2008) – Borhuetter-Ferguson method.

Given: the data of the run-off triangle $\quad y_{ij}, \quad i, j = 0,\dots,t, \quad i + j \le t$

the initial estimate of expected ultimate claims $\quad \mu_0^{(i)}, \quad i = 0,\dots,t$

we set

$$\psi_{U,i} = E(U_i) = \mu_0^{(i)}, \quad i = 0,\dots,t; \qquad \psi_{V,i+j} = E(V_{i+j}) = 1, \quad i + j = 0, \dots, 2t$$

We get the quasi-HGLM estimates of the

fixed effect parameters $\quad \hat{\beta}_j, \; j = 0,\dots,t$

random effects $\quad \hat{w}_{U,i} = \log(\hat{u}_i), \; i = 0,\dots,t; \quad \hat{w}_{V,i+j} = \log(\hat{v}_{i+j}) \; i + j \le t$

dispersion parameters $\quad \hat{\phi}, \quad \hat{\lambda}_U \quad$ and $\quad \hat{\lambda}_V$

I+j	0	1	2	3	4	5	6	7	8	9
$\hat{v}_{i+j}$	0.98	1.10	1.09	1.04	1.01	1.01	0.96	0.96	0.94	0.92

$\widehat{CoV}_V \cong 7\%$

A Poisson-gamma HGLM with diagonal effects

Origin year i	Initial estimate $\mu_0^{(i)}/1000$	$\hat{u}_i/1000$	Reserve	RMSEP	RMSEP %
1	11,367	11,271	16,389	20,295	1.238
2	10,963	11,064	27,841	24,917	0.895
3	10,617	10,654	38,434	27,926	0.727
4	11,045	11,063	96,297	41,488	0.431
5	11,481	11,497	176,998	54,905	0.310
6	11,414	11,392	332,200	73,887	0.222
7	11,127	10,943	540,715	93,593	0.173
8	10,987	10,894	1,213,470	146,811	0.121
9	11,618	11,665	4,291,646	355,320	0.083
Total			6,733,989	521,451	0.077

Model	Reserve	RMSEP	RMSEP %
HGLM without diagonal effect	6,235,486	419,505	0.067
CL-ODP	6,047,061	429,891	0.071

GRAZIE PER L'ATTENZIONE!