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**VALUTAZIONE DELLA RISERVA SINISTRI:
DAL METODO DELLA CATENA AGLI HGLM**

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VALUTAZIONE DELLA RISERVA SINISTRI: DAL METODO DELLA CATENA AGLI HGLM

Overview

- Introduction
- Chain-ladder model
- Conjugate HGLM, HGLM and quasi-HGLM
- Bornhuetter-Ferguson model
- Overdispersed Poisson-gamma model
- HGLM estimates for credible claims reserves
- A Poisson-gamma HGLM with diagonal effects

INTRODUCTION

t the maximum development year
 available information at the end of year t

$$y_{ij} : i, j = 0, \dots, t, \quad i + j \leq t$$

- probabilistic assumptions for the **incremental payments**

$$\{ Y_{ij} \mid i, j = 0, \dots, t \}$$

	Development years						
Origin years	0	1		j			t
0							
1							
i	Y_{i0}			Y_{ij}	Y_{it-i}		Y_{it}
					unknown		
t							

run-off table

- **Predictions** of future payments: $Y_{ij} : i + j > t$; **prediction error evaluations**

- Claims reserve for origin year i

$$R_i = \sum_{j=t-i+1}^t Y_{ij}, \quad i = 1, \dots, t$$

- Total claims reserve

$$R = \sum_{i=1}^t R_i = \sum_{i,j:i+j>t} Y_{ij}$$

CHAIN-LADDER MODEL

CL algorithm

observations $c_{ij} = \sum_{k=0}^j y_{ik}$ **cumulative claims**

CL estimates for the **link ratios** (development factors):

$$\hat{f}_j = \frac{\sum_{i=0}^{t-j-1} c_{i,j+1}}{\sum_{i=0}^{t-j-1} c_{ij}} \quad j = 0, \dots, t-1$$

	0		j	$j+1$			t
0							
1							
$t-j-1$							
t							

CL estimate for the ultimate claims C_{it} : $\hat{C}_{it}^{CL} = c_{i,t-i} \prod_{j=t-i}^{t-1} \hat{f}_j$

CL estimates for the claims reserve: $\hat{R}_i^{CL} = \hat{C}_{it}^{CL} - c_{i,t-i}$

Poisson model

Y_{ij} independent, Poisson distributed with

$$E(Y_{ij}) = a_i \gamma_j \quad \text{for all } i = 0, \dots, t, \quad j = 0, \dots, t \quad \text{and} \quad \sum_{j=0}^t \gamma_j = 1$$

- $E(C_{it}) = a_i \quad i = 0, \dots, t$ expected value of the ultimate claims for origin year i
- $\gamma_j \quad j = 0, \dots, t$ expected “cash-flow pattern” (rate of the ultimate claims paid in development period j)

Let the **maximum likelihood (ML) estimates** of the parameters of the Poisson model

$$\hat{a}_i^{POI}, \quad i = 0, \dots, t, \quad \hat{\gamma}_j^{POI}, \quad j = 0, \dots, t$$

Poisson ML estimate for the ultimate claims: $\hat{C}_{it}^{POI} = c_{i,t-i} + \hat{a}_i^{POI} \sum_{j=t-i+1}^t \hat{\gamma}_j^{POI}$

Poisson ML estimate for the claims reserve: $\hat{R}_{it}^{POI} = \hat{a}_i^{POI} \sum_{j=t-i+1}^t \hat{\gamma}_j^{POI}$

Remark: $\hat{C}_{it}^{POI} = \hat{C}_{it}^{CL} \quad \hat{R}_{it}^{POI} = \hat{R}_{it}^{CL}$

Remark:

➤ Poisson model:

- the incremental payments Y_{ij} are independent Poisson distributed
- $E(Y_{ij}) = a_i \gamma_j \Leftrightarrow \log(E(Y_{ij})) = \log(a_i) + \log(\gamma_j)$ for all $i = 0, \dots, t, j = 0, \dots, t$

➤ **Generalized Linear Model** (quasi-GLM, Overdispersed Poisson)

a1) **Independence assumption**

$$Y_{ij} \text{ independent} \quad i = 0, \dots, t, \quad j = 0, \dots, t$$

a2) **Distributional assumption**

$$Y_{ij} \sim EDF \quad E(Y_{ij}) = \mu_{ij}, \quad \text{var}(Y_{ij}) = \phi V(\mu_{ij})$$

a3) **Structural assumption**

$$g(\mu_{ij}) = \mathbf{x}_{ij}^T \boldsymbol{\beta}$$

CONJUGATE HGLM, HGLM AND QUASI-HGLM

Lee, Nelder, Pawitan (2006), Generalized linear models with random effects: a unified approach via h-likelihood

A claims reserving model in the HGLM framework

Let

Y_{ij} response variables, incremental payments $i, j = 0, 1, \dots, t$

U_i **unobservable risk parameters** (origin years) $i = 0, \dots, t$

x_{ij} vector of covariates

Model assumptions (Conjugate HGLM)

a1) *Independence assumptions*

U_0, \dots, U_t independent

$Y_{ij}|U = \mathbf{u}$, $i = 0, \dots, t$, $j = 0, \dots, t$, independent for any value \mathbf{u} of $U = (U_0, \dots, U_t)$

$$(Y_{ij}|U = \mathbf{u}) \stackrel{d}{=} (Y_{ij}|U_i = u_i)$$

a2) *Distributional assumptions for the responses conditional on risk parameters*

$$(Y_{ij}|U_i = u_i) \sim EDF$$

$$E(Y_{ij}|U_i = u_i) = b'(\theta_{ij}) = \mu_{ij}, \quad \text{var}(Y_{ij}|U_i = u_i) = \frac{\phi}{\omega_{ij}} V(\mu_{ij})$$

with ω_{ij} , $i, j = 0, \dots, t$, known weights (e.g. exposure measure)

a3) **Structural assumptions**

$$\mu_{ij} = g^{-1}(\mathbf{x}_{ij}^T \boldsymbol{\beta} + w_i)$$

where $g = b'^{-1}$ (canonical link)

$\boldsymbol{\beta}$ **fixed effects parameters**

$w_i = g(u_i)$ **random effects**

a4) **Distributional assumptions for the risk parameters**

$W_i = g(U_i) = b'^{-1}(U_i)$ distribution conjugate of the EDF of $Y_{ij}|U_i = u_i$,

$$f_{W_i}(w) = \exp\left\{\frac{1}{\lambda_i}[\psi_i w - b(w)]\right\} d(\psi_i, \lambda_i)$$

where ψ_i and λ_i are **hyperparameters** (λ_i is the **dispersion parameter**)

➤ $\psi_i = E(U_i)$

➤ $\lambda_i = \text{var}(U_i) / E(U_i^p)$ in Tweedie models (where $V(\mu) = \mu^p$ for some p)

(Jewell (1974), Bühlmann, Gisler (2005), Ohlsson, Johansson (2006))

Parameter estimation:

Hierarchical log-likelihood or ***h-likelihood*** (Lee, Nelder (1996))

$$h \equiv \log f_{Y,W} = l_{Y|W=w} + l_W$$

where $l_{Y|W=w}$ is the log-likelihood of $Y|W = w$

l_W is the logarithm of the density of W

Given the data y and the hyperparameters ψ , if the dispersion parameters ϕ and λ are known, the maximum h-likelihood estimates of the fixed effect parameters and the random effects are the solution of

$$\begin{cases} \partial h / \partial \boldsymbol{\beta} = \mathbf{0} \\ \partial h / \partial \mathbf{w} = \mathbf{0} \end{cases}$$

with
$$h(\boldsymbol{\beta}, \mathbf{w}; \phi, \boldsymbol{\lambda}; \mathbf{y}, \boldsymbol{\psi}, \boldsymbol{\omega}) = \sum_{\substack{i,j \\ i+j \leq t}} \frac{\omega_{ij}}{\phi} \left[y_{ij} (\mathbf{x}_{ij}^T \boldsymbol{\beta} + w_i) - b(\mathbf{x}_{ij}^T \boldsymbol{\beta} + w_i) \right] + \sum_{i=0}^t \frac{\omega_i}{\phi} \left[\psi_i w_i - b(w_i) \right]$$

where $\theta_{ij} = \mathbf{x}_{ij}^T \boldsymbol{\beta} + w_i$ because of the canonical link; $\omega_i = \phi / \lambda_i$.

The *h-likelihood* $h(\boldsymbol{\beta}, \mathbf{w}; \phi, \boldsymbol{\lambda}; \mathbf{y}, \boldsymbol{\psi}, \boldsymbol{\omega})$ can be viewed as the log-likelihood of an ***augmented GLM***.

HGLM AND QUASI-HGLM

Conjugate HGLM can be extended.

HGLM

- The link function g is not necessarily the canonical one
- transform of the risk parameter in the linear predictor $w_i = g_1(u_i)$; not necessarily $g = g_1$
- $W_i = g_1(U_i)$, follows a distribution conjugate of an EDF with cumulant b_1 (or variance function V_1); not necessarily $b_1 = b$ (or $V_1 = V$).

Quasi-HGLM

Provides an extension similar to that leading from GLMs to quasi-GLMs:

- no full distributional assumptions are necessary
- the dispersion parameters ϕ e λ_i can be estimated through a regression structure:

$$\phi_{ij} = g_{\phi}^{-1}(\mathbf{x}_{\phi,ij}^T \boldsymbol{\gamma}_{\phi}), \quad \lambda_i = g_{\lambda}^{-1}(\mathbf{x}_{\lambda,i}^T \boldsymbol{\gamma}_{\lambda}).$$

A quasi-HGLM can be fitted by **estimating iteratively three interconnected GLMs or quasi-GLMs.**

BORNHUETTER-FERGUSON MODEL

Let

$E(C_{it}) = \mu_i \quad i = 0, \dots, t$ expected value of the ultimate claims

$0 < b_0 \leq b_1 \leq \dots \leq b_t = 1$ expected claims development pattern

b_j rate of the ultimate claims expected to be paid within development year j

BF estimate for the ultimate claims C_{it} : $\hat{C}_{it}^{BF} = c_{i,t-i} + \hat{\mu}_i(1 - \hat{b}_{t-i})$

BF estimates for the claims reserve: $\hat{R}_{it}^{BF} = \hat{\mu}_i(1 - \hat{b}_{t-i})$

where $\hat{\mu}_i$ “initial” estimates of the ultimate claims

$\hat{b}_{t-i} = \prod_{j=t-i}^{t-1} \hat{\lambda}_j^{-1}$ CL estimate of the development pattern

In fact, $\hat{C}_{it}^{CL} = c_{i,t-i} \prod_{j=t-i}^{t-1} \hat{f}_j$ $c_{i,t-i} = \hat{C}_{it}^{CL} (\prod_{j=t-i}^{t-1} \hat{\lambda}_j)^{-1} = \hat{C}_{it}^{CL} \hat{b}_{t-i}$

$$\hat{R}_{it}^{CL} = \hat{C}_{it}^{CL} - \hat{C}_{it}^{CL} \hat{b}_{t-i} = \hat{C}_{it}^{CL} (1 - \hat{b}_{t-i}) = \frac{c_{i,t-i}}{\hat{b}_{t-i}} (1 - \hat{b}_{t-i})$$

The two “extreme position” can be combined \Rightarrow **CREDIBLE CLAIMS RESERVES**

OVERDISPERSED POISSON-GAMMA MODEL

Let

$\{ Y_{ij} \mid i, j = 0, \dots, t \}$ hierarchical overdispersed Poisson (ODP) - gamma
(England, Verrall (2002); Verrall (2004); Wüthrich (2007))

with

- Y_{ij} , $i, j = 0, \dots, t$, *response variables* (incremental payments)
- $U = (U_0, \dots, U_t)$ **unobservable risk parameters** related to the origin years,

1) **Independence assumptions**

U_0, \dots, U_t are independent,

$Y_{ij} \mid U = \mathbf{u}$, $i, j = 0, \dots, t$, are independent for any value \mathbf{u} of U ,

$$(Y_{i0}, \dots, Y_{it}) \mid U = \mathbf{u} \stackrel{d}{=} (Y_{i0}, \dots, Y_{it}) \mid U_i = u_i$$

2) **Distributional assumptions for the responses conditional on risk parameters**

$$Y_{ij}|U_i = u_i \sim \text{ODP}, \quad E(Y_{ij}|U_i = u_i) = \mu_{ij}, \quad \text{Var}(Y_{ij}|U_i = u_i) = \phi\mu_{ij}$$

3) **Structural assumption**

$$\mu_{ij} = \exp(\log(u_i) + \beta_j) = u_i \exp(\beta_j)$$

4) **Distributional assumptions for the risk parameters**

$$U_i \sim \text{Gamma}, \quad E(U_i) = \psi_i, \quad \text{var}(U_i) / E(U_i) = \lambda_i$$

where ψ_i and λ_i **hyperparameters**; λ_i **dispersion parameter**

- $E(Y_{ij}|U_i = u_i) = u_i \exp(\beta_j) = \exp(\beta_j + w_i)$, with $w_i = \log(u_i)$
- $W_i = \log(U_i)$ follows a distribution conjugate of the ODP of $Y_{ij}|U_i = u_i$,
- The assumptions 1), 2), 3), 4) define a quasi-**HGLM**
with β_j , **fixed effects**, w_i , **random effects**

Overdispersed Poisson-gamma model

Some well-known results (e.g. Bühlmann, Gisler (2005); Verrall (2004); Wüthrich (2007))

Given the parameters $\beta_j, j = 0, \dots, t$, and ϕ
 the hyperparameters ψ_i and $\lambda_i, i = 0, \dots, t$

$$\mathcal{D}_t = \{ Y_{ij} : i + j \leq t \}$$

$$\Rightarrow E(U_i | \mathcal{D}_t) = z_i \frac{\sum_{j=0}^{t-i} Y_{ij}}{\sum_{j=0}^{t-i} \exp(\beta_j)} + (1 - z_i) \psi_i \quad \text{where} \quad z_i = \frac{\sum_{j=0}^{t-i} \exp(\beta_j)}{\sum_{j=0}^{t-i} \exp(\beta_j) + \frac{\phi}{\lambda_i}};$$

$$\Rightarrow E(R_i | \mathcal{D}_t) = \left[z_i \frac{\sum_{j=0}^{t-i} Y_{ij}}{\sum_{j=0}^{t-i} \exp(\beta_j)} + (1 - z_i) \psi_i \right] \sum_{j=t-i+1}^t \exp(\beta_j) \quad \text{Bayesian estimator of } R_i$$

Overdispersed Poisson-gamma model

We have (e.g. England, Verrall (2002); Wüthrich (2007)):

$$E(R_i | \mathcal{D}_t) = \left[z_i \frac{C_{i,t-i}}{b_{t-i}} + (1 - z_i) E(C_{it}) \right] (1 - b_{t-i})$$

with $b_j = \frac{\sum_{h=0}^j \exp(\beta_h)}{\sum_{h=0}^t \exp(\beta_h)}$, $j = 0, \dots, t$ the claims development pattern; $E(C_{it}) = \psi_i \sum_{h=0}^t \exp(\beta_h)$

$$\Rightarrow E(R_i | \mathcal{D}_t) = z_i \tilde{R}_{it}^{BCL} + (1 - z_i) \tilde{R}_{it}^{BBF}$$

Mixture of CL and BF estimators:

	Bayesian estimators
CL model (Mack(1993))	$\tilde{R}_i^{BCL} = \frac{C_{i,t-i}}{b_{t-i}} (1 - b_{t-i})$
BF model (Mack (2008); Wüthrich, Merz (2008))	$\tilde{R}_i^{BBF} = \mu_i (1 - b_{t-i})$

if, in the ODP-Gamma, CL and BF models, the parameters are such that the claims development pattern b_j , $j = 0, \dots, t$ is the same and $E(C_{it}) = \mu_i$.

HGLM ESTIMATES FOR CREDIBLE CLAIMS RESERVES

Given: the data of the run-off triangle y_{ij} , $i, j = 0, \dots, t$, $i + j \leq t$

the “external data”, initial estimate of ultimate claims $\mu_0^{(i)}$, $i = 0, \dots, t$

We set $\psi_i = E(U_i) = \mu_0^{(i)}$, $i = 0, \dots, t$

We estimate

the fixed effect parameters β_j , $j = 0, \dots, t$

the random effects $w_i = \log(u_i)$, $i = 0, \dots, t$

the dispersion parameters ϕ and λ_i , $i = 0, \dots, t$

HGLM estimates for credible claims reserves

Given the estimates of the dispersion parameters, the quasi-HGLM estimates of the fixed and random effects satisfy the following conditions:

$$\left\{ \begin{array}{l} \exp(\hat{\beta}_j) = \frac{\sum_{i=0}^{t-j} y_{ij}}{\sum_{i=0}^{t-j} \hat{u}_i} \quad j = 0, \dots, t \\ \hat{u}_i = \hat{z}_i \frac{\sum_{j=0}^{t-i} y_{ij}}{\sum_{j=0}^{t-i} \exp(\hat{\beta}_j)} + (1 - \hat{z}_i) \psi_i \quad i = 0, \dots, t, \end{array} \right. \quad \text{with } \hat{z}_i = \frac{\sum_{j=0}^{t-i} \exp(\hat{\beta}_j)}{\sum_{j=0}^{t-i} \exp(\hat{\beta}_j) + \frac{\hat{\phi}}{\hat{\lambda}_i}}$$

Compare with

$$E(U_i | \mathcal{D}_t) = z_i \frac{\sum_{j=0}^{t-i} Y_{ij}}{\sum_{j=0}^{t-i} \exp(\beta_j)} + (1 - z_i) \psi_i \quad z_i = \frac{\sum_{j=0}^{t-i} \exp(\beta_j)}{\sum_{j=0}^{t-i} \exp(\beta_j) + \frac{\phi}{\lambda_i}}$$

Prediction

For the claims reserve $R = \sum_{i=1}^t R_i = \sum_{i,j:i+j>t} Y_{ij}$ we have

$$\bar{R} = E(R|U) = \sum_{i,j:i+j>t} \exp(\beta_j + W_i)$$

and consider the estimator

$$\tilde{R} = \sum_{i,j:i+j>t} \exp(\tilde{\beta}_j + \tilde{w}_i)$$

where $\tilde{\beta}$ and \tilde{w} are the quasi-HGLM estimators of β and W .

HGLM estimates of the claims reserves

$$\hat{R}_i = \hat{u}_i \sum_{j=t-i+1}^t \exp(\hat{\beta}_j) = \left[\hat{z}_i \frac{\sum_{j=0}^{t-i} y_{ij}}{\sum_{j=0}^{t-i} \exp(\hat{\beta}_j)} + (1 - \hat{z}_i) \psi_i \right] \sum_{j=t-i+1}^t \exp(\hat{\beta}_j)$$

$$= \left[\hat{z}_i \frac{\sum_{j=0}^{t-i} y_{ij}}{\hat{b}_{t-i}} + (1 - \hat{z}_i) \psi_i \sum_{j=0}^t \exp(\hat{\beta}_j) \right] (1 - \hat{b}_{t-i}) = \hat{z}_i \hat{R}_i^{BCL} + (1 - \hat{z}_i) \hat{R}_i^{BBF}$$

where

- $\hat{b}_j = \frac{\sum_{h=0}^j \exp(\hat{\beta}_h)}{\sum_{h=0}^t \exp(\hat{\beta}_h)}, \quad j = 0, \dots, t$
- $\hat{R}_i^{BCL} = \frac{\sum_{j=0}^{t-i} y_{ij}}{\hat{b}_{t-i}} (1 - \hat{b}_{t-i}), \quad \hat{R}_i^{BBF} = \left(\psi_i \sum_{j=0}^t \exp(\hat{\beta}_j) \right) (1 - \hat{b}_{t-i})$
- $\psi_i \sum_{j=0}^t \exp(\hat{\beta}_j)$ is an estimate of $E(C_{it}) = \psi_i \sum_{h=0}^t \exp(\beta_h)$

Conditional mean square error of prediction

$$\begin{aligned} \text{MSEP}_{R|\mathcal{D}_t}(\tilde{R}) &= E\left[(R - \tilde{R})^2 | \mathcal{D}_t\right] \\ &= E\left[\text{Var}(R|U) | \mathcal{D}_t\right] + \text{var}(\bar{R} | \mathcal{D}_t) + E\left[\left(E(\bar{R} | \mathcal{D}_t) - \tilde{R}\right)^2 | \mathcal{D}_t\right] \end{aligned}$$

Following Booth, Hobert (1998) and Lee, Ha (2009), estimates of the three terms are:

- $\hat{E}\left[\text{Var}(R|U) | \mathcal{D}_t\right] = \sum_{i,j:i+j>t} \hat{\phi} \exp(\hat{\beta}_j + \hat{w}_i)$
- $\hat{\text{var}}(\bar{R} | \mathcal{D}_t) = \hat{\text{var}}(r(\mathbf{W}) | \mathcal{D}_t) \approx J_r(\mathbf{w}) \mathbf{H}_{22}^{-1} J_r(\mathbf{w})^T |_{\hat{\delta}}$
- $\hat{E}\left[\left(E(\bar{R} | \mathcal{D}_t) - \tilde{R}\right)^2 | \mathcal{D}_t\right] \approx \hat{E}\left[\left(f(\boldsymbol{\beta}) - f(\tilde{\boldsymbol{\beta}})\right)^2 | \mathcal{D}_t\right] \approx J_f(\boldsymbol{\beta}) \mathbf{G}^{-1} J_f(\boldsymbol{\beta})^T |_{\hat{\delta}}$

where

- $J_r(\mathbf{w})$ and $J_f(\boldsymbol{\beta})$ the Jacobian matrices of the functions

$$r(\mathbf{w}) = \sum_{i,j:i+j>t} \exp(\beta_j + w_i) \quad f(\boldsymbol{\beta}) = \sum_{i,j:i+j>t} g^{-1}(\mathbf{x}_{ij}^T \boldsymbol{\beta} + \tilde{w}_i(\boldsymbol{\beta}))$$

with $\tilde{w}_i(\boldsymbol{\beta})$ the HGLM estimator of w_i for a given $\boldsymbol{\beta}$

$$- \mathcal{I}(\boldsymbol{\delta}) = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{12}^T & \mathbf{H}_{22} \end{bmatrix}, \quad \mathcal{I}(\boldsymbol{\delta})^{-1} = \begin{bmatrix} \mathbf{G}^{-1} & \mathbf{F} \\ \mathbf{F}^T & \mathbf{C} \end{bmatrix}$$

Remark:

The quasi-HGLM is fitted by **estimating iteratively three interconnected quasi-GLMs** and the first one is an augmented GLM for the run-off data y and the pseudo-data $\boldsymbol{\psi}$

At convergence, the inverse of the Fisher information matrix $\mathcal{I}(\hat{\boldsymbol{\delta}})^{-1}$, $\boldsymbol{\delta} = (\boldsymbol{\beta}^T, \mathbf{w}^T)^T$, of this augmented GLM provides an estimate of the **variance-covariance matrix**

$$\text{var} \begin{pmatrix} \tilde{\boldsymbol{\beta}} \\ \tilde{\mathbf{w}} - \mathbf{W} \end{pmatrix}$$

EXAMPLE

Data: Tables 2.4-2.5 in Wüthrich, Merz (2008) – Borhuetter-Ferguson method.

Example: ODP-Gamma, quasi-HGLM model with $\phi_{ij} = \phi$ and $\lambda_i = \lambda$.

Origin year i	Initial estimate ψ_i /1000	Quasi-HGLM		
		\hat{u}_i /1000	$\sum_{j>t-i} \exp(\hat{\beta}_j)$ × 100	\hat{R}_i
1	11,367	11,906	0.1277	15,199
2	10,963	11,799	0.2214	26,125
3	10,617	10,952	0.3183	34,857
4	11,045	11,159	0.7762	86,623
5	11,481	11,459	1.3909	159,377
6	11,414	11,006	2.6763	294,565
7	11,127	10,219	4.6062	470,703
8	10,987	10,190	10.6646	1,086,682
9	11,618	11,194	36.2808	4,061,355

HGLM estimates for credible claims reserves

$$\hat{R}_i = \hat{z}_i \hat{R}_i^{BCL} + (1 - \hat{z}_i) \hat{R}_i^{BBF}$$

Origin year i	\hat{R}_i	Quasi-HGLM				\hat{R}_i^{CL}	\hat{R}_i^{BF}	$1 - \hat{b}_{t-i}^{CL}$
		\hat{z}_i	\hat{R}_i^{BCL}	\hat{R}_i^{BBF}	$1 - \hat{b}_{t-i}$			
1	15,199	0.7391	15,442	14,511	0.00145	15,125	16,124	0.00142
2	26,125	0.7389	26,780	24,274	0.00251	26,257	26,998	0.00246
3	34,857	0.7387	35,234	33,791	0.00361	34,538	37,575	0.00354
4	86,623	0.7377	86,939	85,734	0.00880	85,301	95,434	0.00864
5	159,377	0.7363	159,268	159,682	0.01578	156,494	178,023	0.01551
6	294,565	0.7334	290,603	305,462	0.03036	286,121	341,306	0.02990
7	470,703	0.7289	455,156	512,508	0.05225	449,167	574,090	0.05160
8	1,086,682	0.7138	1,052,603	1,171,674	0.12097	1,043,243	1,318,646	0.12002
9	4,061,355	0.6254	3,969,176	4,215,257	0.41154	3,950,815	4,768,385	0.41042

Remark: $\hat{R}_i^{BBF} = \left[\mu_0^{(i)} \sum_{j=0}^t \exp(\hat{\beta}_j) \right] (1 - \hat{b}_{t-i})$, $\hat{R}_i^{BF} = \mu_0^{(i)} (1 - \hat{b}_{t-i}^{CL})$, $\sum_{j=0}^t \exp(\hat{\beta}_j) = 0.88159$

$$\hat{\phi} = 14,895$$

$$\hat{\lambda} = 47,936$$

$$\hat{\phi} / \hat{\lambda} = 0.31$$

HGLM estimates for credible claims reserves

Model	Reserve	RMSEP	RMSEP %
HGLM	6,235,486	419,505	0.067
CL-ODP	6,047,061	429,891	0.071
BF(Alai <i>et al.</i> (2009))	7,356,575	471,971	0.064
Bühlmann-Straub(Wüthrich,Merz(2008))	6,159,709	317,540	0.052
Bayesian(England <i>et al.</i> (2012)) - $a=100$	6,145,526	422,526	0.069
Bayesian(England <i>et al.</i> (2012)) - $a \rightarrow \infty$	6,664,047	395,012	0.059

	HGLM	CL-ODP
$\hat{\phi}$	14,895	14,714

A POISSON-GAMMA HGLM WITH DIAGONAL EFFECTS

Let

Y_{ij} response variables, incremental payments $i, j = 0, 1, \dots, t$

$(\mathbf{U}, \mathbf{V}) = (U_0, \dots, U_t, V_0, \dots, V_{2t})$ **unobservable risk parameters**

$U_i, \quad i = 0, \dots, t,$ is related to the origin year i

$V_{i+j} \quad i, j = 0, \dots, t$ is related to the calendar year or payment year $i + j$

a1) **Independence assumptions**

The risk parameters $(\mathbf{U}, \mathbf{V}) = (U_0, \dots, U_t, V_0, \dots, V_{2t})$ are independent,

Conditionally on (\mathbf{U}, \mathbf{V}) , Y_{ij} , $i, j = 0, \dots, t$, are independent,

$$\left[Y_{ij} | (\mathbf{U}, \mathbf{V}) = (\mathbf{u}, \mathbf{v}) \right] \stackrel{d}{=} \left[Y_{ij} | (U_i, V_{i+j}) = (u_i, v_{i+j}) \right], \quad i, j = 0, \dots, t, \text{ for any } (\mathbf{u}, \mathbf{v})$$

a2) **Distributional assumptions for the responses, conditional on the risk parameters**

$[Y_{ij}|(U_i, V_{i+j}) = (u_i, v_{i+j})]$ are **overdispersed Poisson** distributed

$$E[Y_{ij}|(U_i, V_{i+j}) = (u_i, v_{i+j})] = \mu_{ij} \quad \text{var}[Y_{ij}|(U_i, V_{i+j}) = (u_i, v_{i+j})] = \phi_{ij} \mu_{ij}$$

a3) **Structural assumptions for the expectations**

$$E[Y_{ij}|(U_i, V_{i+j}) = (u_i, v_{i+j})] = \mu_{ij} = \exp(\beta_j + w_{U,i} + w_{V,i+j}) = e^{\beta_j} u_i v_{i+j}$$

where β_j regression parameter (fixed effect) for development year j

$w_{U,i} = \log(u_i)$ and $w_{V,i+j} = \log(v_{i+j})$ are the random effects

a4) **Distributional assumptions for the risk parameters**

Let $W_{U,i} = \log(U_i)$ $W_{V,i+j} = \log(V_{i+j})$

with $U_i \sim \text{Gamma}$, $E(U_i) = \psi_{U,i}$, $\text{var}(U_i) / E(U_i) = \lambda_{U,i}$

with $V_{i+j} \sim \text{Gamma}$, $E(V_{i+j}) = \psi_{V,i+j}$, $\text{var}(V_{i+j}) / E(V_{i+j}) = \lambda_{V,i+j}$

where $\psi_{U,i}$, $\psi_{V,i+j}$, $\lambda_{U,i}$ and $\lambda_{V,i+j}$ **hyperparameters;**
 $\lambda_{U,i}$ and $\lambda_{V,i+j}$ **dispersion parameter**

To estimate the dispersion parameters ϕ_{ij} , $\lambda_{U,i}$, $\lambda_{V,i+j}$, $i, j = 0, \dots, t$, we make the following assumptions

a5) **Structural assumptions for the dispersions**

$$\phi_{ij} = \phi, \quad i, j = 0, \dots, t,$$

$$\lambda_{U,i} = \lambda_U, \quad i = 0, \dots, t, \quad \lambda_{V,i+j} = \lambda_V, \quad i + j = 0, \dots, 2t$$

We estimate a quasi-Hierarchical Generalized Linear Model

Remark:

The maximum h-likelihood estimates of the fixed effect parameters and the random effects are the solution of

$$\begin{cases} \partial h / \partial \boldsymbol{\beta} = \mathbf{0} \\ \partial h / \partial \mathbf{w} = \mathbf{0} \end{cases}$$

We get

$$\hat{w}_{V,i+j} = \log(\psi_{V,i+j}) \quad \text{for } i + j > t$$

Prediction

For the claims reserve

$$R = \sum_{i=1}^t R_i = \sum_{i,j:i+j>t} Y_{ij}$$

we consider the estimator

$$\tilde{R} = \sum_{i,j:i+j>t} \exp(\tilde{\beta}_j + \tilde{w}_{U,i} + \tilde{w}_{V,i+j})$$

HGLM estimate of the claims reserve

$$\begin{aligned}\hat{R}_i &= \hat{u}_i \sum_{j=t-i+1}^t \exp(\hat{\beta}_j) \hat{v}_{i+j} \\ &= \left(\hat{z}_{U,i} \frac{\sum_{j=0}^{t-i} y_{ij} \hat{v}_{i+j}}{\sum_{j=0}^{t-i} \exp(\hat{\beta}_j) \hat{v}_{i+j}} + (1 - \hat{z}_{U,i}) \psi_{U,i} \right) \sum_{j=t-i+1}^t \exp(\hat{\beta}_j) \hat{v}_{i+j}\end{aligned}$$

Conditional mean square error of prediction

$$MSEP_{R|\mathcal{D}_t}(\tilde{R}) = E\left[(R - \tilde{R})^2 | \mathcal{D}_t\right] = E[\text{var}(R|U, V) | \mathcal{D}_t] + \text{var}(E(R|U, V) | \mathcal{D}_t) + (E(R | \mathcal{D}_t) - \tilde{R})^2$$

can be estimated in a similar way as before.

EXAMPLE

Data: Tables 2.4-2.5 in Wüthrich, Merz (2008) – Borhuetter-Ferguson method.

Given: the data of the run-off triangle y_{ij} , $i, j = 0, \dots, t$, $i + j \leq t$

the initial estimate of expected ultimate claims $\mu_0^{(i)}$, $i = 0, \dots, t$

we set

$$\psi_{U,i} = E(U_i) = \mu_0^{(i)}, \quad i = 0, \dots, t; \quad \psi_{V,i+j} = E(V_{i+j}) = 1, \quad i + j = 0, \dots, 2t$$

We get the quasi-HGLM estimates of the

fixed effect parameters $\hat{\beta}_j$, $j = 0, \dots, t$

random effects $\hat{w}_{U,i} = \log(\hat{u}_i)$, $i = 0, \dots, t$; $\hat{w}_{V,i+j} = \log(\hat{v}_{i+j})$ $i + j \leq t$

dispersion parameters $\hat{\phi}$, $\hat{\lambda}_U$ and $\hat{\lambda}_V$

$l+j$	0	1	2	3	4	5	6	7	8	9
\hat{v}_{i+j}	0.98	1.10	1.09	1.04	1.01	1.01	0.96	0.96	0.94	0.92

$$\widehat{CoV}_V \cong 7\%$$

A Poisson-gamma HGLM with diagonal effects

Origin year i	Initial estimate $\mu_0^{(i)}/1000$	$\hat{u}_i/1000$	Reserve	RMSEP	RMSEP %
1	11,367	11,271	16,389	20,295	1.238
2	10,963	11,064	27,841	24,917	0.895
3	10,617	10,654	38,434	27,926	0.727
4	11,045	11,063	96,297	41,488	0.431
5	11,481	11,497	176,998	54,905	0.310
6	11,414	11,392	332,200	73,887	0.222
7	11,127	10,943	540,715	93,593	0.173
8	10,987	10,894	1,213,470	146,811	0.121
9	11,618	11,665	4,291,646	355,320	0.083
Total			6,733,989	521,451	0.077

Model	Reserve	RMSEP	RMSEP %
HGLM without diagonal effect	6,235,486	419,505	0.067
CL-ODP	6,047,061	429,891	0.071

GRAZIE PER L'ATTENZIONE!