Sensitivity Analysis of Annuity Models

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Introduction

- Annuities are among the most important life insurance products.
- The cost of annuities is determined by mathematical models based on financial and demographic factors.
- For adequate risk management actions it becomes necessary an effective uncertainty quantification of factor risks: which risk has the biggest impact in determining the cost of annuities? Are there completely irrelevant factors? Do they interact?
- We present a comprehensive framework for Sensitivity Analysis (SA) of annuities based at different scales.

Annuity model

The annuity model we consider is the standard whole-life continuous annuity at age x

$$\bar{a}_x = \int_0^\infty {}_t p_x \exp\left[-\delta t\right] dt, \tag{1}$$

where δ represents the force of interest. The surviving probability is given by ${}_t p_x = \exp\left[-\int_0^t \mu_{x+s} ds\right]$, where μ_x is the force of mortality at age x.

We assume that force of mortality (at time 0) at age $x + u$ follows the Gompertz law with parameters b and c

$$\mu^0_{x+u} = \exp[b + c(x + u)]. \tag{2}$$

and that the mortality rates decrease by an exponential reduction function of the form $\exp[-\alpha t]$, so that

$$\mu^t_{x+u} = \mu^0_{x+u} e^{-\alpha t}. \tag{3}$$

Under these assumptions, the probability of surviving t years at age x (on a cohort basis) becomes

$${}_t p_x = \exp\left[-\mu^0_x \left(\frac{e^{(c-\alpha)t} - 1}{c - \alpha}\right)\right]. \tag{4}$$

Consequently, the cost of annuity becomes the function

$$\bar{a}\left(\mu_x^0, c, \alpha, \delta\right) = \int_0^\infty \exp\left[-\mu_x^0\left(\frac{e^{(c-\alpha)t}-1}{c-\alpha}\right)\right] e^{-\delta t} dt \qquad (5)$$

with $\alpha < c$.

A first very simple way to investigate the model is to evaluate it when inputs vary one factor at a time from a base-case input $\mathbf{x}^0 = \left(\mu_x^0, c, \alpha, \delta\right)^0$ to a best case $\mathbf{x}^+ = \left(\mu_x^0, c, \alpha, \delta\right)^+$ and to a worst case $\mathbf{x}^- = \left(\mu_x^0, c, \alpha, \delta\right)^-$.

Local Sensitivity analysis: Finite Changes

In the general case, denote with $g(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$ the input-output mapping of interest. Then, we can define the finite-change sensitivity measures [Borgonovo and Plischke, 2016]

$$\Delta_i^+ y = g(x_i^+ : \mathbf{x}_{-i}^0) - g(\mathbf{x}^0) \tag{6}$$

and

$$\Delta_i^- y = g(x_i^- : \mathbf{x}_{-i}^0) - g(\mathbf{x}^0) \tag{7}$$

where $(x_i^+ : \mathbf{x}_{-i}^0)$ and $(x_i^- : \mathbf{x}_{-i}^0)$ denote the scenario in which the $i-$th input is changed according to the best or worst case, respectively, for $i = 1, ..., n$.

Finite change decomposition

For any multivariate mapping it is possible to decompose the finite change $\Delta g = g(\mathbf{x}^1) - g(\mathbf{x}^0)$ across two different scenarios $\mathbf{x}^0$ and $\mathbf{x}^1$ with the finite-change ANOVA expansion [Borgonovo, 2010]

$$\Delta g = \sum_{i=1}^{n} \Delta g_i + \sum_{i<j} \Delta g_{i,j} + \sum_{i<j<k} \Delta g_{i,j,k} + \ldots + \Delta g_{1,2,\ldots,n}, \quad (8)$$

where the $2^n - 1$ finite change effects of increasing dimension are recursively given by

$$\begin{cases} \Delta g_i = g(x_i^1 : \mathbf{x}_{-i}^0) - g(\mathbf{x}^0) \\ \Delta g_{i,j} = g(x_{i,j}^1 : \mathbf{x}_{-i,j}^0) - \Delta g_i - \Delta g_j - g(\mathbf{x}^0) \\ \ldots \end{cases} \quad (9)$$

The effects Δg_i, $i = 1, 2, \ldots, n$, are called main effects and the other higher order terms are the interaction effects.

Total finite-change indices

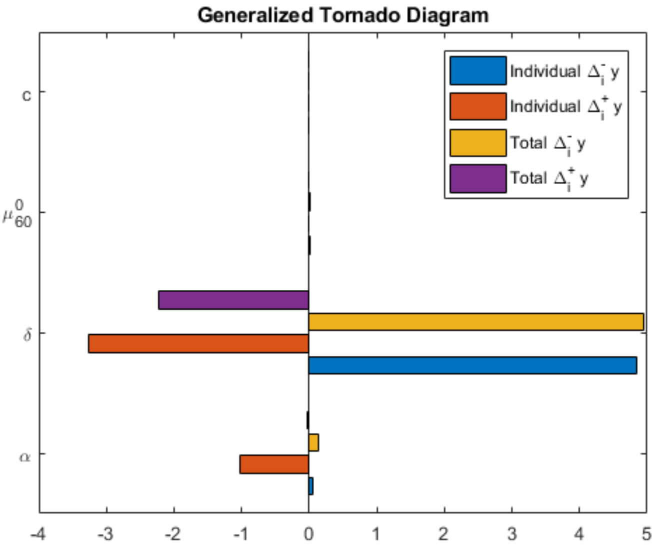
In general, given the decomposition (8), it is possible to define the total effect of factor x_i

$$\Delta_i^T g = \Delta g_i + \sum_{j=1, j\neq i}^{n} \Delta g_{i,j} + \sum_{k,j=1, k\neq i\neq j}^{n} \Delta g_{i,j,k} + \ldots + \Delta g_{1,2,\ldots,n} \quad (10)$$

which is a measure of the total impact of x_i to the total change Δg. Analogously, the total interaction effect is

$$\Delta_i^I g = \Delta_i^T g - \Delta g_i \quad (11)$$

the difference between the total and the main effects of x_i.



Generalized Tornado Diagram
Individual Δ_i^- y
Individual Δ_i^+ y
Total Δ_i^- y
Total Δ_i^+ y
c
μ_60^0
δ
α
-4
-3
-2
-1
0
1
2
3
4
5

From Local to Global Sensitivity Analysis

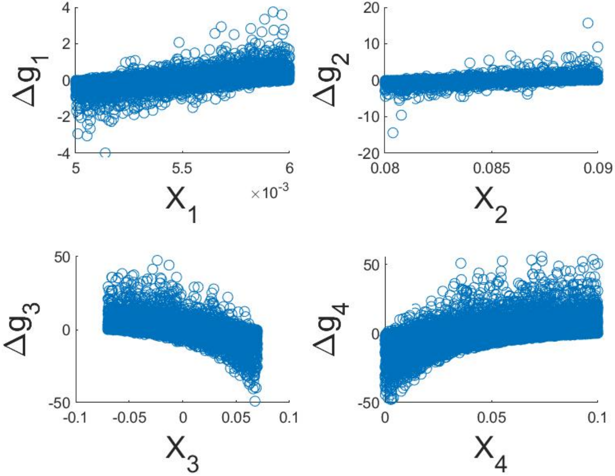
- Local sensitivity methods provide insights only around the base point $\mathbf{x}^0$.
- Inputs have typically a range and global measures become of interest.
- To explore the whole input space one can consider a series of two scenarios sampled across all the input space and then aggregate local importance measures.
- Method of Elementary Effects [Morris, 1991; Campolongo et al., 2011; Borgonovo and Rabitti, ?].

Input space

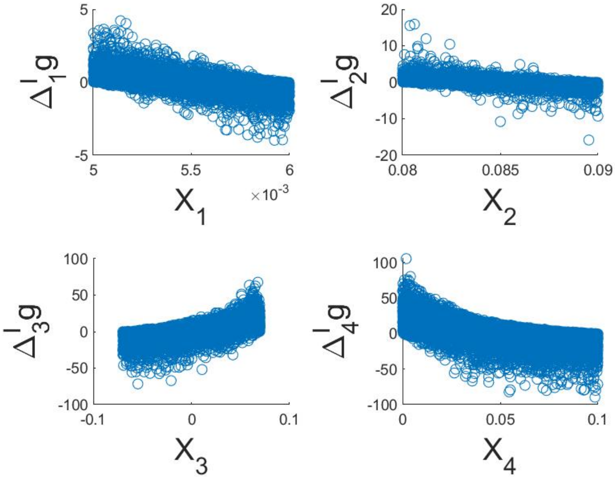
Parameter	Estimated value	Lower range	Upper range
μ_{60}^{0}	0.00552155	0.005	0.006
c	0.085	0.08	0.09
α	-	-0.07	0.07
δ	-	0%	10%

Parameter values and ranges from [Haberman et al., 2011]. They are estimated by regression using the Continuous Mortality Investigation (1991-1994) mortality table for female pensioners at ages 60 and over. [Haberman et al., 2011] introduce ranges of variation for the parameters c, α and δ. We have chosen the range of μ_{60}^{0}.

Scatterplots with $N = 10000$



Scatterplots with $N = 10000$



GSA: Pearson correlation coefficient

[Pearson, 1905] defines the correlation coefficient

$$\eta_i^2 = \frac{Cov(Y, X_i)}{\sigma_Y \sigma_{X_i}} \tag{12}$$

where σ_{X_i} is the standard deviation of the input X_i, $i = 1, ..., n$. This index measures the linear dependence between the two variables Y and X_i.

GSA: Functional ANOVA expansion

[Hoeffding, 1948; Efron and Stein, 1981] prove that the multivariate mapping g can be decomposed as

$$g(\mathbf{x}) = g_0 + \sum_{i=1}^{n} g_i(x_i) + \sum_{i<j} g_{i,j}(x_i, x_j) \ldots + g_{1,2,\ldots,n}(x_1, x_2, \ldots, x_n) \tag{13}$$

where

$$\begin{cases} g_0 = \int g(\mathbf{x}) d\mu_{\mathbf{X}} \\ g_i(x_i) = \int g(\mathbf{x}) d\mu_{\mathbf{X}_{-i}} - g_0 \\ g_{i,j}(x_i, x_j) = \int g(\mathbf{x}) d\mu_{\mathbf{X}_{-i,j}} - g_i(x_i) - g_j(x_j) - g_0 \\ \ldots \end{cases} . \tag{14}$$

GSA: Sobol' indices - 1

Under independence the terms $g_z(x_z)$, $z \subseteq \{1, ..., n\}$, are orthogonal.
The output variance $\sigma^2_{\mathbf{Y}}$ can be decomposed as

$$\sigma^2_{\mathbf{Y}} = \sum_{i=1}^{n} \sigma^2_i + \sum_{i<j} \sigma^2_{i,j} ... + \sigma^2_{1,2,...,n} \tag{15}$$

where $\sigma^2_z = V[g_z(x_z)]$ is the variance of the group of variables indexed by $z \subseteq \{1, ..., n\}$. Every term can be interpreted as

$$\sigma^2_z = Var_{X_z}\left[E_{X_{-z}}[Y|X_z]\right]. \tag{16}$$

The index (16) has been used by [Bruno et al. 2000; Karabey et al. 2014] to study the risk of a portfolio of life insurance policies with mortality and interest rate risks.

GSA: Sobol' indices - 2

If we normalize by the total variance, one finds

$$\sum_{i=1}^{n} S_i + \sum_{i<j} S_{i,j\ldots} + S_{1,2,\ldots,n} = 1, \tag{17}$$

where the generic term is the sensitivity index of [Sobol', 1993] and is given by

$$S_z = \frac{\sigma_z^2}{\sigma_{\mathbf{Y}}^2}. \tag{18}$$

Every term S_z measures the proportion of the output variance which the inputs x_z contribute to.

Total Sobol' effects

[Homma and Saltelli, 1996] define the total effect of the inputs x_z as

$$S_z^T = \sum_{u \cap z \neq \emptyset} S_u. \tag{19}$$

Itis a measure of the total impact of inputs in z.
The sensitivity measures S_z and S_z^T can shed light on the importance of the inputs z in explaining the output variability.

Moment-independent sensitivity methods

[Baucells and Borgonovo, 2013] consider the sensitivity index β_i^{KS}

$$\beta_i^{KS} = E\left[\sup_y |F_{\mathbf{Y}}(y) - F_{\mathbf{Y}|X_i}(y)|\right]. \tag{20}$$

Suppose now that the output admits a density $f_{\mathbf{Y}}(y)$. [Borgonovo, 2007] defines the δ_i^{BO} sensitivity measure

$$\delta_i^{BO} = \frac{1}{2} E\left[\int |f_{\mathbf{Y}}(y) - f_{\mathbf{Y}|X_i}(y)| dy\right]. \tag{21}$$

These sensitivity measures are invariant under monotonic transformations.

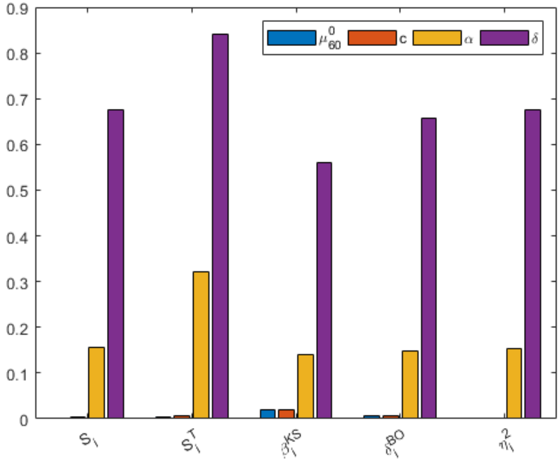
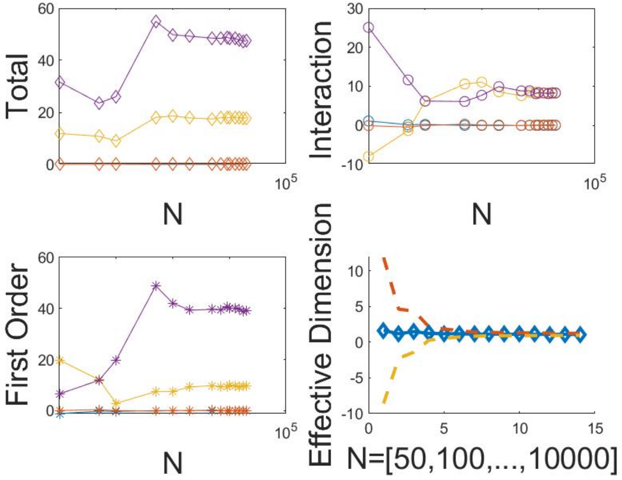


Figure 1: Sensitivity indices estimated from $N = 10000$ Monte Carlo runs.



Total
N
Interaction
N
First Order
N
Effective Dimension
N=[50,100,...,10000]

SA of annuities with dependent financial and mortality factors

[Deelstra et al., 2016; Dacorogna and Apicella, 2016] consider the role of dependence between mortality and interest rate in actuarial valuations.
However, in such case there are some theoretical complications to calculate the variance-based indices [Li and Rabitz, 2017].
Nonetheless, moment-independent measures can still be computed.

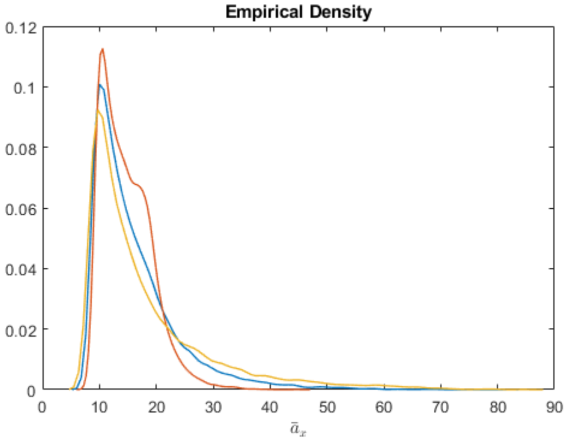


Figure 2: The empirical density of the annuity model for independent (blue line), positively correlated (red line) and negatively correlated inputs (yellow line).

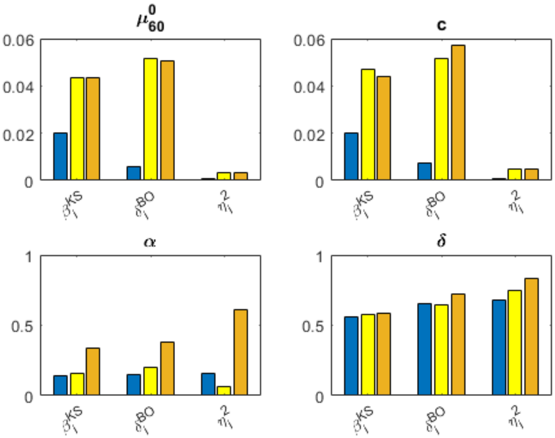


Figure 3: Comparison of moment-independent sensitivity measures in absence of correlation (blue bars) and with positive (yellow bars) and negative correlation (golden bars) of 0.6 between α and δ. The Monte Carlo runs are $N = 10000$.

Conclusions

- In the past it has been debated whether financial risk connected to life annuities is more important than the mortality risk.
- We have proposed the comprehensive framework of [Borgonovo, Plischke and Rabitti, submitted] to investigate the importance of these factors in determining the cost of annuities.
- Our results in the global case are in line with those of [Karabey et al., 2014]. Moreover, we also provide insights on the local and global scale with dependence.
- Future research: SA for stochastic simulation for portfolios of variable annuities.