Optimal annuitisations after retirement

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Aim and content

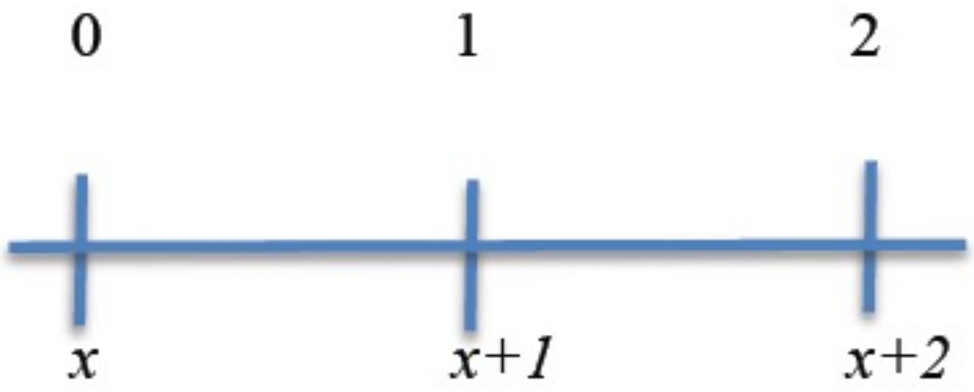
- The reforms in public pension schemes have progressively reduced post retirement social security income.
- Individuals join pension funds and individual plans to increase their wealth at retirement. These types of plans give the possibility to retire the capital accumulated into the scheme or to convert into annuity.
- The research analyses post retirement choices.
- An optimisation model is applied to determine the expected utility.
- Bequest motive is taken into account.

Preliminary remarks

- Yaari (1965) proposes the optimisation life cycle model.
- Fischer (1973) presents a life cycle model for life insurance product.
- Walliser (1999) discusses the consumption in old age. He analyses different kind of product and simulate the evolution of the consumption also in the case of bequeathable wealth.
- Vidal-Melià et al. (2002) present different types of individual capitalization systems and decisions for the consumer in retirement.
- Lockwood (2012) analyses bequest motive and annuity puzzle. The results is that bequest motive plays a central role in limiting the demand for annuities.
- In this research we extend the analysis of Walliser and Vidal for post retirement consumption allocation.

A two-period model

- **Assumptions**
- The individual has access only to annuity and bond markets.
- The individual can purchase an annuity at age x in exchange of a single premium.
- He receives the annuity benefit at age x and, if he is alive, at age $x+1$ and he dies at $x+2$.
- W_0 is the initial wealth.
- α is the fraction of the initial wealth invested into annuity.



Wealth evolution

- Wealth

- W_t is the wealth at time t
- $0 \rightarrow \; W_0$
- $1 \rightarrow \; W_1 = (W_0 - P_0 + B_0 - C_0) \cdot e^{R_f}$
- $2 \rightarrow \; W_2 = (W_1 + B_1 - C_1) \cdot e^{R_f}$

- α Is the fraction of the initial wealth to be annuitised
- B_t is the benefit received at time t
- $\ddot{a}_x$ is the price of an anticipated annuity
- M is the insurance charge annual rate
- $i =$ technical rate used by the insurer

- C_t is the consumption at time t
- R_f is the risk free annual rate
- P_0 is the premium paid at time 0 ;
- $P_0 = \alpha \cdot W_0$
- $B_0 = \ldots = B_t = \frac{P_0}{\ddot{a}_x} = \frac{\alpha W_0}{\ddot{a}_x}$
- $\ddot{a}_x = \frac{1}{1-M} \sum_{t=0}^{T-1} \frac{1}{(1+i)^t} \, {}^{i}p_x^t$
- ${}^{i}p_x^t$ is the survival probability for people aged x at time t used by the insurer

General optimisation problem

- $$\max_{C,\,0\le\alpha\le1} \sum_{t=0}^{T-1} \frac{U(C_t)}{(1+\delta)^t} p_x^t$$

$$s.t.\, W_1 = \left[W_0 - P_0 + B_0 - C_0\right]\cdot\left[e^{R_f}\right]$$

$$\vdots$$

$$W_{t+1} = \left[W_t + B_t - C_t\right]\cdot\left[e^{R_f}\right],$$

$$P_0 = \alpha \cdot W_0$$

$$B_t = \frac{P_0}{\ddot{a}_x}$$

$$\ddot{a}_x = \frac{1}{1-M} \sum_{t=0}^{T-1} \frac{1}{(1+i)^t}\, {}^{i}p_x^t$$

- δ is the classical discount factor future utility rate of time preference
- p_x^t is the real survival probability for people aged x at time t.

Optimisation problem for $T=2$

- $$\max_{C_0, C_1, 0\le\alpha\le1}\left[U(C_0)+\frac{U(C_1)}{(1+\delta)}\,{}_{1}p_x\right]$$
  $$s.t.\ W_1=\left[W_0-P_0+B_0-C_0\right]\cdot e^{R_f}$$
  $$s.t.\ W_2=\left[W_1+B_1-C_1\right]\cdot e^{R_f}$$
  $$P_0=\alpha\cdot W_0$$
  $$B_t=\frac{P_0}{\ddot{a}_x}$$
  $$\ddot{a}_x=\frac{1}{1-M}\sum_{t=0}^{T-1}\frac{1}{(1+i)^t}\,{}^{i}p_x^{t}$$

- In absence of bequest we add the condition $W_2=0$.
- By the two constraints:
- $$C_0+C_1\cdot e^{-R_f}=(1-\alpha)\,W_0+\left(1+e^{-R_f}\right)\cdot\alpha\frac{W_0}{\ddot{a}_x}$$
- Net present value of consumption of the two periods can't be different from net present value of total wealth.

Utility function

- CRRA Utility function

$$U(C_t) = \begin{cases} \dfrac{C_t^{1-\gamma} - 1}{1-\gamma} \text{ where } \gamma \neq 1 \\ \ln C_t, \text{ where } \gamma = 1 \end{cases}$$

- The parameter γ measures the degree of relative risk aversion that is implicit in the utility function.

Solving by Lagrangian

- The Lagrangian is:

$$L = \frac{C_0^{1-\gamma} - 1}{1-\gamma} + \frac{C_1^{1-\gamma} - 1}{1-\gamma}\frac{p_x^1}{1+\delta} + \lambda\left(W_0 - \alpha W_0 + \frac{\alpha W_0}{\ddot{a}_x} + \frac{\alpha W_0}{\ddot{a}_x}\cdot e^{-R_f} - C_0 - C_1 \cdot e^{-R_f}\right)$$

- From the first order condition we obtain:

$$C_1 = C_0\left(\frac{p_x^1}{1+\delta}\cdot e^{R_f}\right)^{\frac{1}{\gamma}}$$

$$C_0 = W_0\frac{1-\alpha+\frac{\alpha}{\ddot{a}_x}+\frac{\alpha}{\ddot{a}_x}\cdot e^{-R_f}}{1+\left(\frac{p_x^1}{1+\delta}\right)^{\frac{1}{\gamma}}\left(e^{R_f}\right)^{\frac{1-\gamma}{\gamma}}}$$

$$C_1 = W_0\frac{1-\alpha+\frac{\alpha}{\ddot{a}_x}+\frac{\alpha}{\ddot{a}_x}\cdot e^{-R_f}}{1+\left(\frac{p_x^1}{1+\delta}\right)^{\frac{1}{\gamma}}\cdot\left(e^{R_f}\right)^{\frac{1-\gamma}{\gamma}}}\left(\frac{p_x^1}{1+\delta}\cdot e^{R_f}\right)^{\frac{1}{\gamma}}$$

Maximisation for α

- We substitute C_0 into the utility function and maximise with respect to α.

$$\xi(\alpha) = \frac{\left[W_0 \frac{1-\alpha+\frac{\alpha}{\ddot{a}_x}+\frac{\alpha}{\ddot{a}_x}\cdot e^{-R_f}}{1+\left(\frac{p_x^1}{1+\delta}\right)^{\frac{1}{\gamma}}\cdot\left(e^{R_f}\right)^{\frac{1-\gamma}{\gamma}}} \right]^{1-\gamma} - 1}{1-\gamma} + \frac{\left[W_0 \frac{1-\alpha+\frac{\alpha}{\ddot{a}_x}+\frac{\alpha}{\ddot{a}_x}\cdot e^{-R_f}}{1+\left(\frac{p_x^1}{1+\delta}\right)^{\frac{1}{\gamma}}\cdot\left(e^{R_f}\right)^{\frac{1-\gamma}{\gamma}}} \left(\frac{p_x^1}{1+\delta}\cdot e^{R_f}\right)^{\frac{1}{\gamma}} \right]^{1-\gamma} - 1}{1-\gamma} \frac{p_x^1}{1+\delta}$$

- The function is not defined when: $\alpha = \dfrac{-1}{-1+\frac{1}{\ddot{a}_x}+\frac{1}{\ddot{a}_x}e^{-R_f}}$

- If γ is not integer, then a power function is defined only for strictly positive value.

Optimal solution

- Analysis of the derivative

1. If $1+e^{-R_f} > \ddot{a}_x$ the sign of the derivative is always positive, the function is increasing.
2. If $1+e^{-R_f} < \ddot{a}_x$ the sign of of the derivative is always negative, the function is decreasing.
3. If $1+e^{-R_f} = \ddot{a}_x$ the derivative is always equal to *0* and the function $\xi(\alpha)$ is a constant.

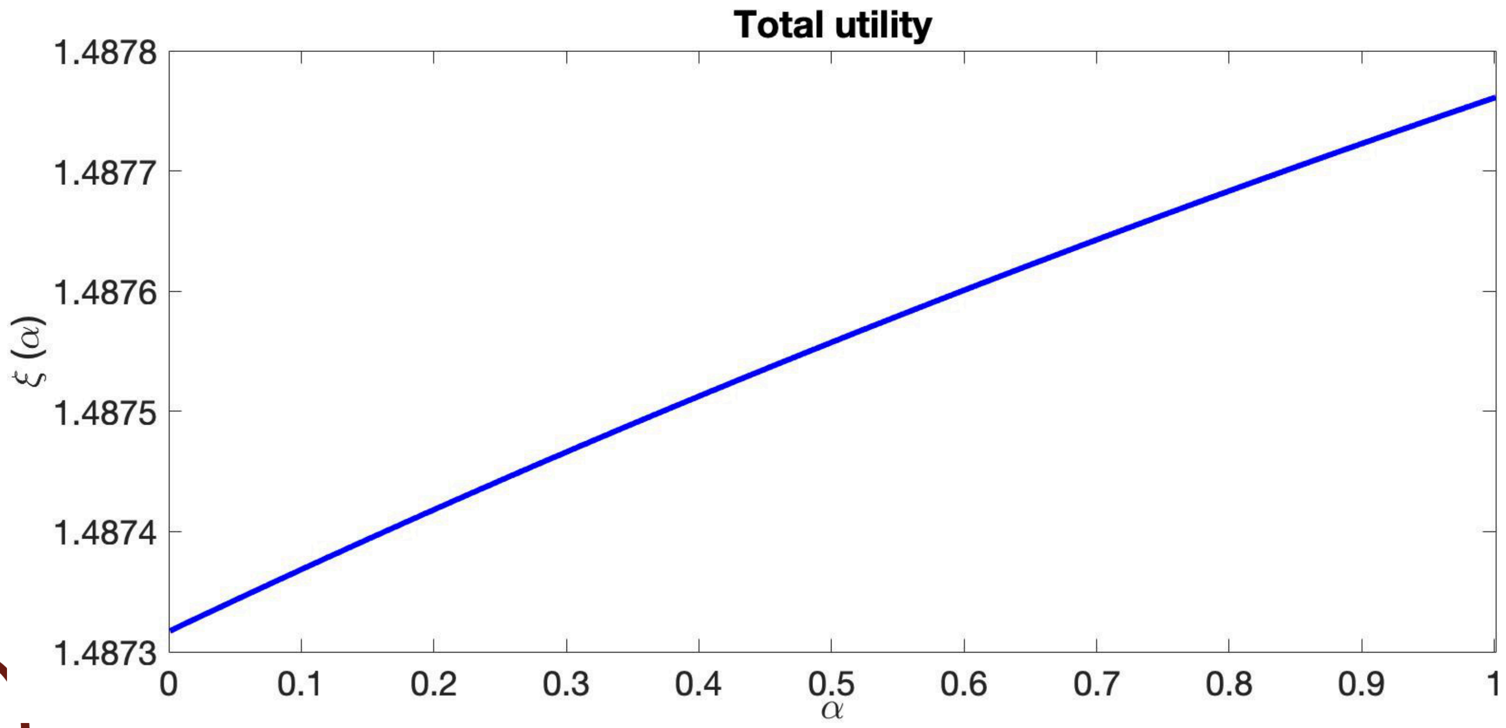
- Optimal solution

1. If $1+e^{-R_f} > \ddot{a}_x$ (normal condition) the optimal solution is to annuitise all the starting wealth: α=1.
2. If $1+e^{-R_f} < \ddot{a}_x$, it will be optimal to invest all the wealth into the financial asset: α=0.

Case 1

- $$1 + e^{-R_f} > \ddot{a}_x$$

W_0	R_f	i	M	γ	δ	p_x^t	${}^i p_x^t$
1000	0.02	0.015	0.05	2	0.02	0.5	0.6

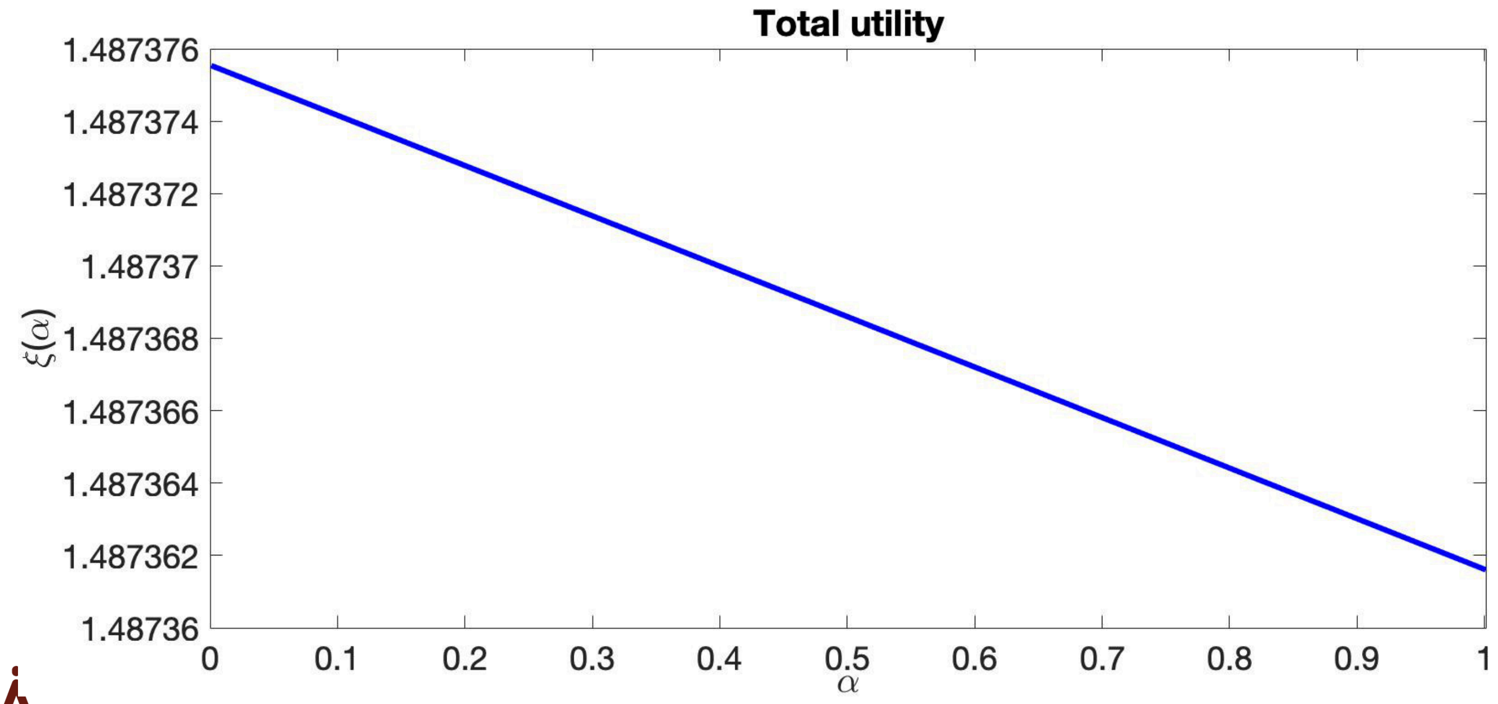


The optimal solution is α=1.

Case 2

$$1 + e^{-R_f} < \ddot{a}_x$$

W_0	R_f	i	M	γ	δ	p_x^t	${}^i p_x^t$
1000	0.06	0.005	0.1	2	0.02	0.5	0.76

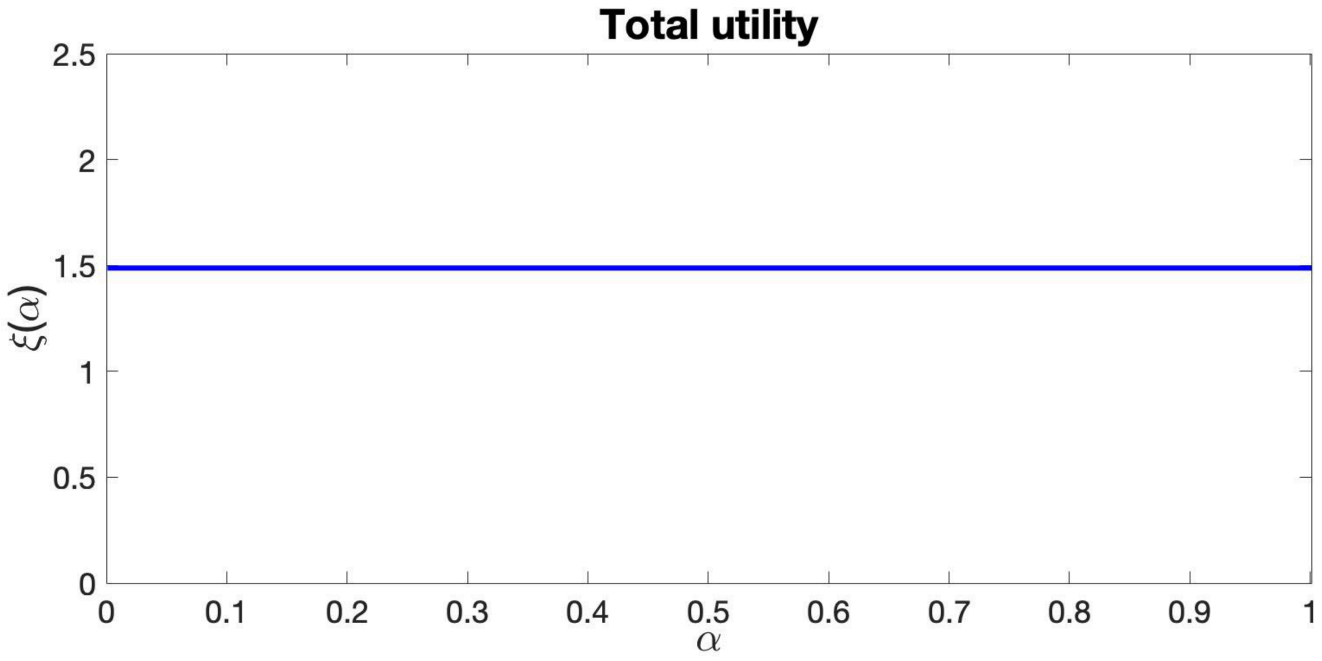


- The optimal solution is α=0.
  - In this case, the annuity price asked by the insurer is too expensive.
  - This explains why it is not optimal to annuitize.

Case 3

- $$1 + e^{-R_f} = \ddot{a}_x$$

W_0	R_f	i	M	γ	δ	p_x^t	${}^i p_x^t$
1000	0.06	0.0015	0.1	2	0.02	0.5	0.7513



- Particular case: the utility is constant.
- Indifference to annuitisation or not.

The multiperiod case

The problem is:

$$\max_{C_t,\,0\le\alpha\le1} \sum_{t=0}^{T-1} \frac{U\left(C_t\right)}{\left(1+\delta\right)^t} p_x^t$$

$$s.t. W_1 = \left[W_0 - \alpha W_0 + \alpha \frac{W_0}{\ddot{a}_x} - C_0\right] \cdot e^{R_f}$$

$$W_{t+1} = \left[W_t + \alpha \frac{W_0}{\ddot{a}_x} - C_t\right] \cdot e^{R_f}$$

$$a_x = \frac{1}{1-M} \sum_{t=0}^{T-1} \frac{1}{\left(1+i\right)^t} p_x^t$$

If the final wealth $W_T=0$

The solution is:

$$L = \sum_{t=0}^{T-1} \left(\frac{C_t^{1-\gamma} - 1}{1-\gamma} \frac{p_x^t}{(1+\delta)^t} \right) + \lambda \left(W_0 - \alpha W_0 + \frac{\alpha W_0}{\ddot{a}_x} \sum_{t=0}^{T-1} \left[e^{R_f}\right]^{-t} - \sum_{t=0}^{T-1} C_t \left[e^{R_f}\right]^{-t} \right)$$

$$C_t = C_{t-1} \left[p_{x+t-1}^1 \frac{e^{R_f}}{1+\delta} \right]^{\frac{1}{\gamma}}$$

$$C_t = C_0 \left[\frac{p_x^t}{\left(1+\delta\right)^t} \left[e^{R_f}\right]^t \right]^{\frac{1}{\gamma}}$$

$$\xi\left(\alpha\right) = \sum_{t=0}^{T-1} \frac{\left[\dfrac{W_0 - \alpha W_0 + \alpha \dfrac{W_0}{\ddot{a}_x} \sum_{t=0}^{T-1} \left[e^{R_f}\right]^{-t}}{\sum_{t=0}^{T-1} \left[\dfrac{p_x^t}{(1+\delta)^t} \left[e^{R_f}\right]^t \right]^{\frac{1}{\gamma}} [e^{R_f}]^{-t}} \left(\dfrac{p_x^t}{(1+\delta)^t} e^{R_f \cdot t} \right)^{\frac{1}{\gamma}} \right]^{1-\gamma} - 1}{1-\gamma} \frac{p_x^t}{\left(1+\delta\right)^t}$$

Optimal solution

- Analysis of the derivative

1. If $\ddot{a}_x < \sum_{t=0}^{T} [e^{R_f}]^{-t}$ the function is increasing.
2. If $\ddot{a}_x > \sum_{t=0}^{T} [e^{R_f}]^{-t}$ the function is decreasing.
3. If $\ddot{a}_x = \sum_{t=0}^{T} [e^{R_f}]^{-t}$ the function is a constant.

- Optimal solution

1. If $\ddot{a}_x < \sum_{t=0}^{T} [e^{R_f}]^{-t}$ (normal condition) the optimal solution is to annuitise all the starting wealth: α=1.
2. If $\ddot{a}_x > \sum_{t=0}^{T} [e^{R_f}]^{-t}$ the optimal solution is to invest all the wealth into the financial asset: α=0.

Case 1

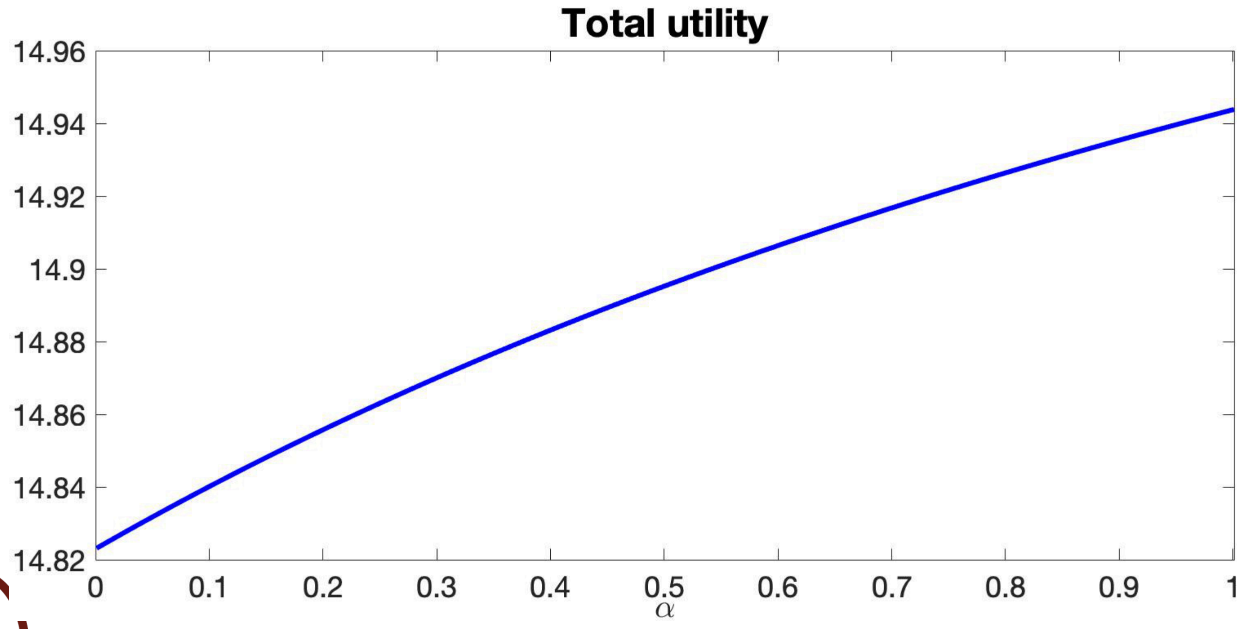
-
- T=46 :

$$\ddot{a}_x < \sum_{t=0}^{T} [e^{R_f}]^{-t}$$

W_0	R_f	i	M	γ	δ
1000	0.02	0.015	0.05	2	0.02

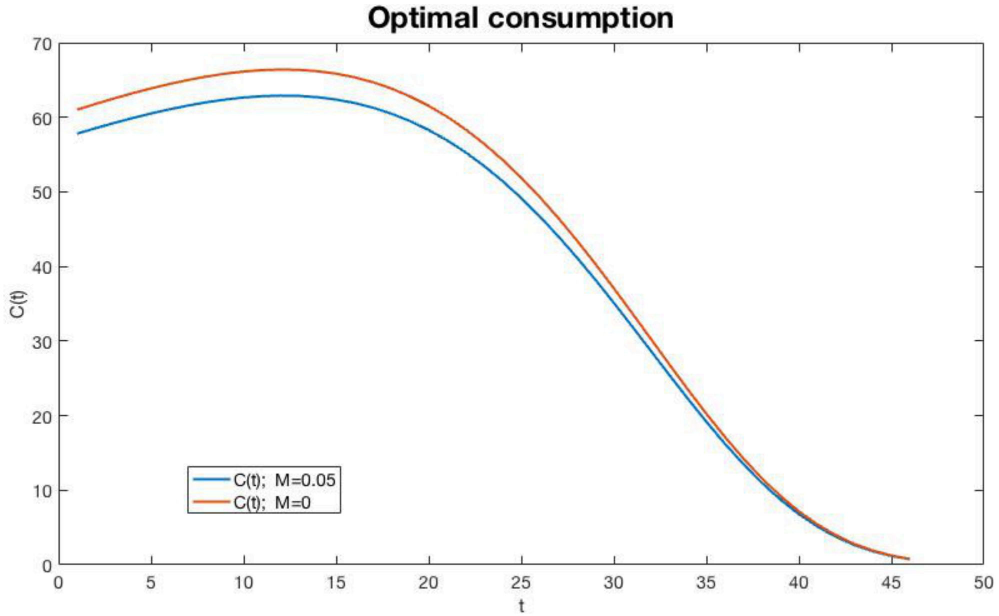
p_x^t: Belgian Mortality tables.

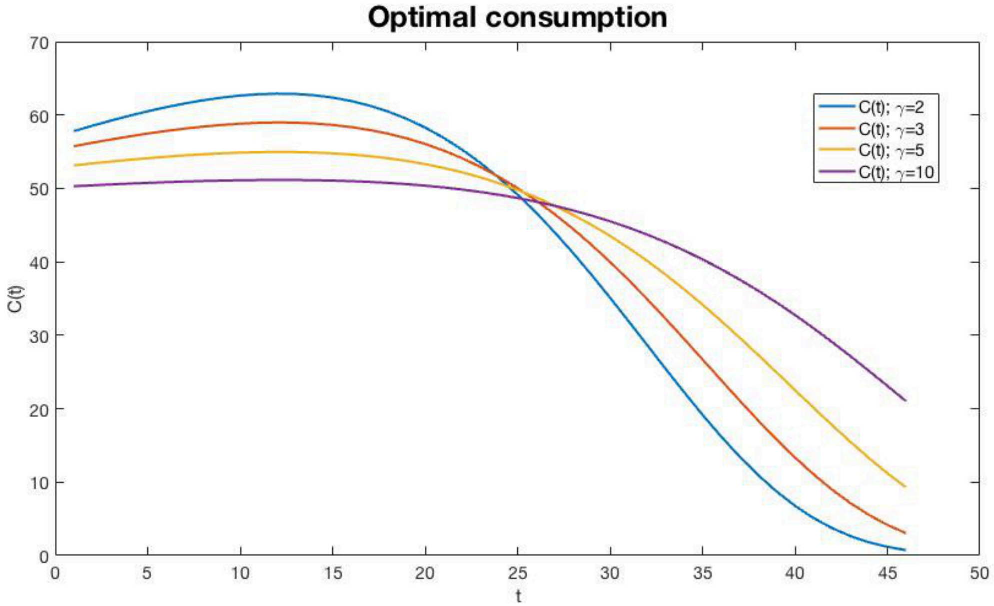
${}^i p_x^t$: Belgian Mortality tables with an age shifting of 5 years.



The optimal solution is α=1.

Optimal consumption: case 1





- Optimal consumption with different survival probability for the insurer.

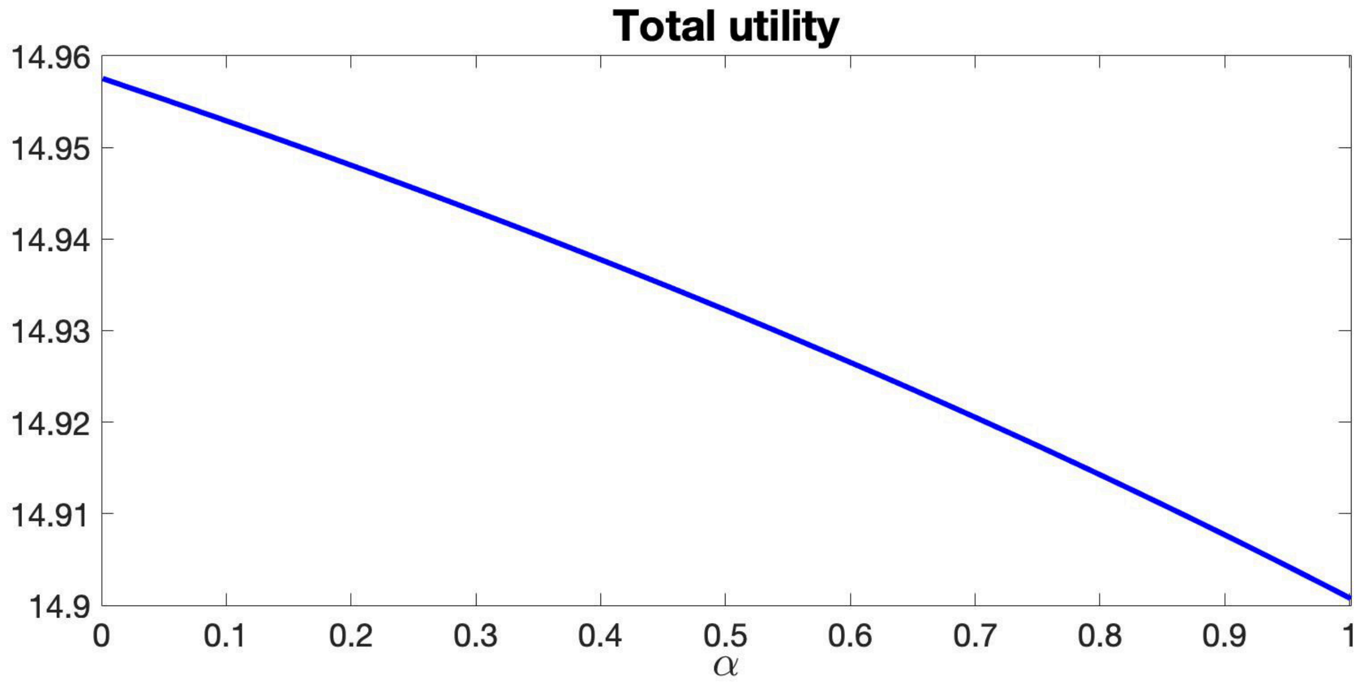
- Optimal consumption with different γ.

Case 2

- $$\ddot{a}_x > \sum_{t=0}^{T} [e^{R_f}]^{-t}$$
- T=46 :

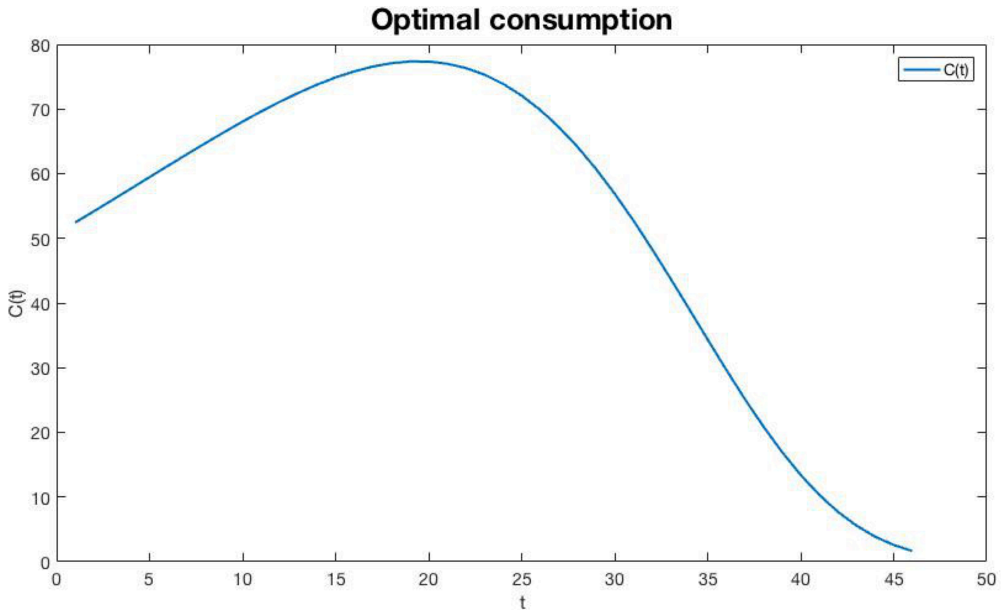
p_x^t: Belgian Mortality tables.

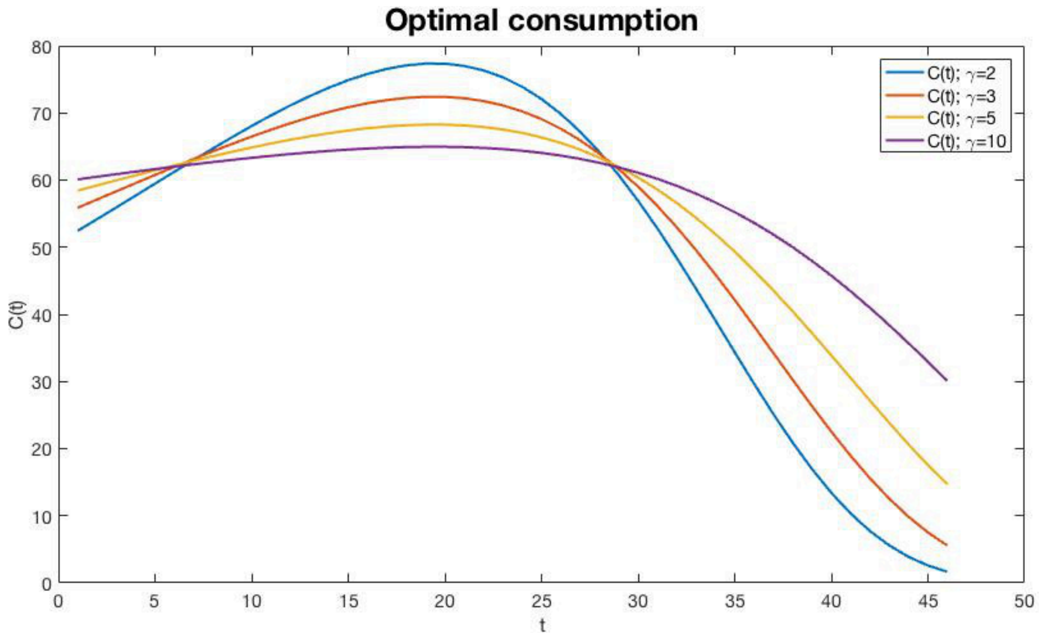
${}^{i}p_x^t$: Belgian Mortality tables with an age shifting of 5 years.



The optimal solution is α=0.

Optimal consumption: case 2



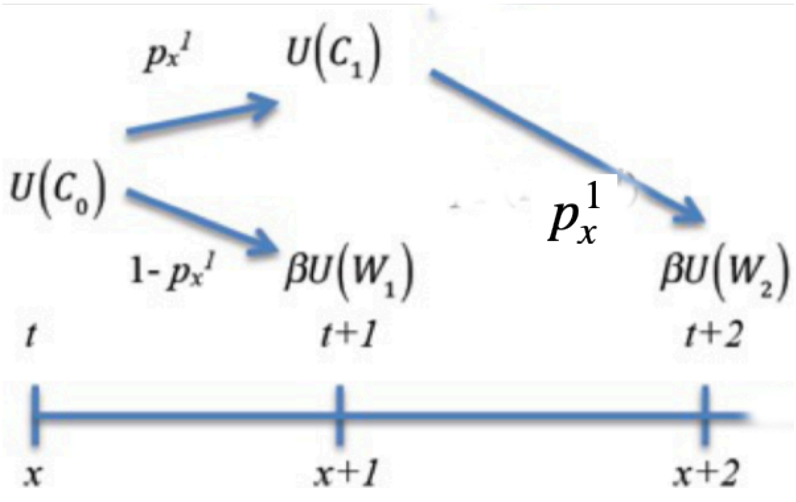


- Optimal consumption with different survival probability for the insurer.
- Being the optimal solution α=0, the optimal consumption path is the same with M=0.

- Optimal consumption with different γ.

Introduction of bequest motive: 2 period model

- With the introduction of bequest, if the individual is not alive at time 1, his heirs will receive the wealth at time 1.
- If he is alive at time 1, he dies at time 2 and his heirs will receive the wealth at time 2.
- β is the bequest coefficient.



$$\max_{C_0, C_1, 0\le\alpha\le1}\left[U\left(C_0\right)+\frac{U\left(C_1\right)}{\left(1+\delta\right)}p_x^1+\beta U\left(W_1\right)\frac{1-p_x^1}{\left(1+\delta\right)}+\beta U\left(W_2\right)\frac{p_x^1}{\left(1+\delta\right)^2}\right]$$

$$s.t. W_1=\left[W_0-P_0+B_0-C_0\right]\cdot\left[e^{R_f}\right]$$

$$W_2=(W_1+B_1-C_1)\cdot e^{R_f}$$

$$P_0=\alpha\cdot W_0 B_t=\frac{P_0}{a_x}$$

$$a_x=\frac{1}{1-M}\sum_{t=0}^{T-1}\frac{1}{\left(1+i\right)^t}p_x^t$$

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Solution

- We consider the same utility function and the same risk aversion parameter for the wealth utility. The Lagrangian is:

$$L = \frac{C_0^{1-\gamma}-1}{1-\gamma} + \frac{C_1^{1-\gamma}-1}{1-\gamma}\frac{p_x^1}{1+\delta} + \beta\frac{W_1^{1-\gamma}-1}{1-\gamma}\cdot\frac{1-p_x^1}{1+\delta} + \beta\frac{W_2^{1-\gamma}-1}{1-\gamma}\cdot\frac{p_x^1}{(1+\delta)^2} + \lambda\left[\left(W_0-\alpha W_0+\frac{\alpha W_0}{a_x}-C_0\right)e^{R_f}-W_1\right] + \mu\left[\left(W_1+\frac{\alpha W_0}{a_x}-C_1\right)e^{R_f}-W_2\right]$$

$$C_1^{-\gamma}\frac{p_x^1}{1+\delta} + \beta W_1^{-\gamma}\frac{1-p_x^1}{1+\delta} = C_0^{-\gamma}\cdot e^{-R_f} \qquad\qquad W_2^{-\gamma}\frac{\beta}{(1+\delta)} = C_1^{-\gamma}\cdot e^{-R_f}$$

- The same result is obtained by applying Euler's condition.
- From this two conditions and the constraints we obtain an implicit equation for C_0.

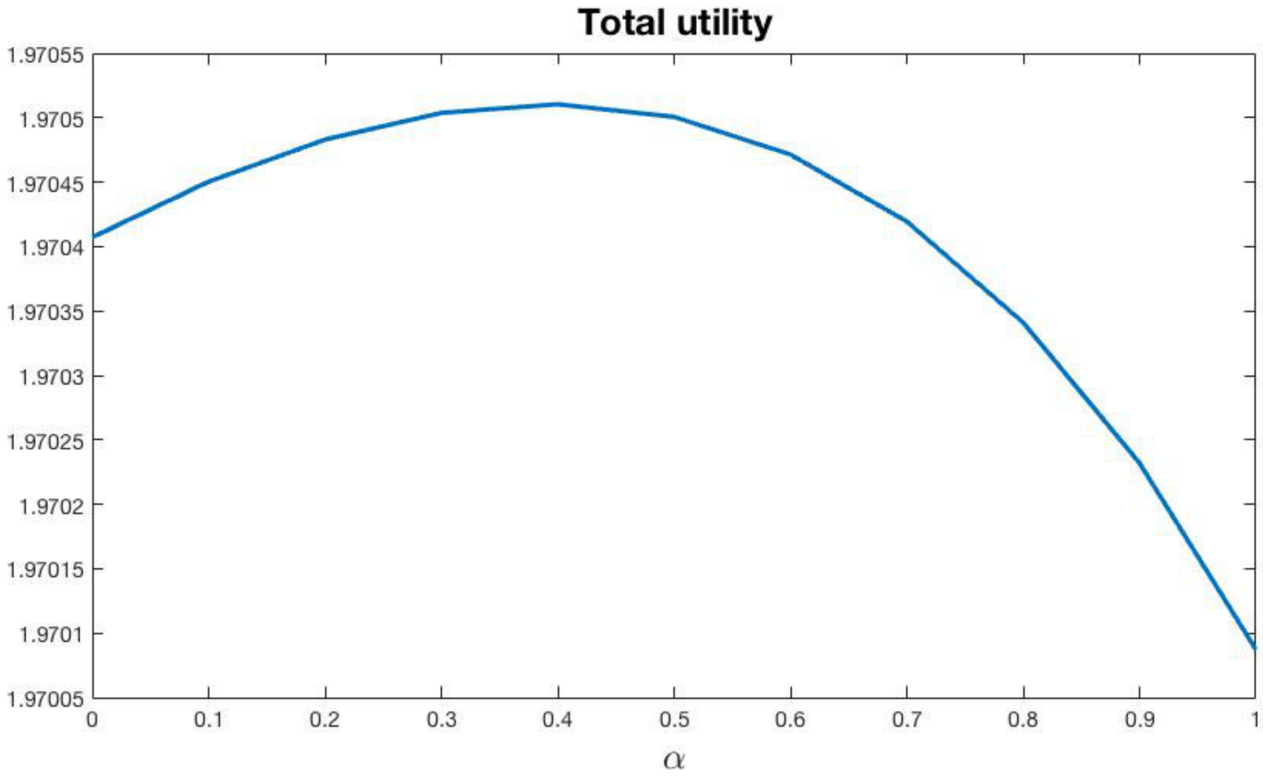
$$C_0 = W_0\left(1-\alpha+\frac{\alpha}{a_x}+\frac{\alpha}{a_x}e^{-R_f}\right) - \left[C_0^{-\gamma}\frac{1+\delta}{p_x^1}e^{-R_f} - \beta\left[W_0\left(1-\alpha+\frac{\alpha}{a_x}\right)-C_0\right]^{-\gamma}\left[e^{R_f}\right]^{-\gamma}\frac{1-p_x^1}{p_x^1}\right]^{-\frac{1}{\gamma}}\cdot e^{-R_f}\left[1+\left(\frac{\beta e^{R_f}}{1+\delta}\right)^{\frac{1}{\gamma}}\cdot\left[e^{-R_f}\right]\right]$$

- When β =0 we come back to the formula without bequest.

Numerical simulation: case 1

$$1 + e^{-R_f} > \ddot{a}_x$$

W_0	β	R_f	i	M	γ	δ	p_x^t	${}^i p_x^t$
1000	0.5	0.02	0.015	0.05	2	0.02	0.5	0.6

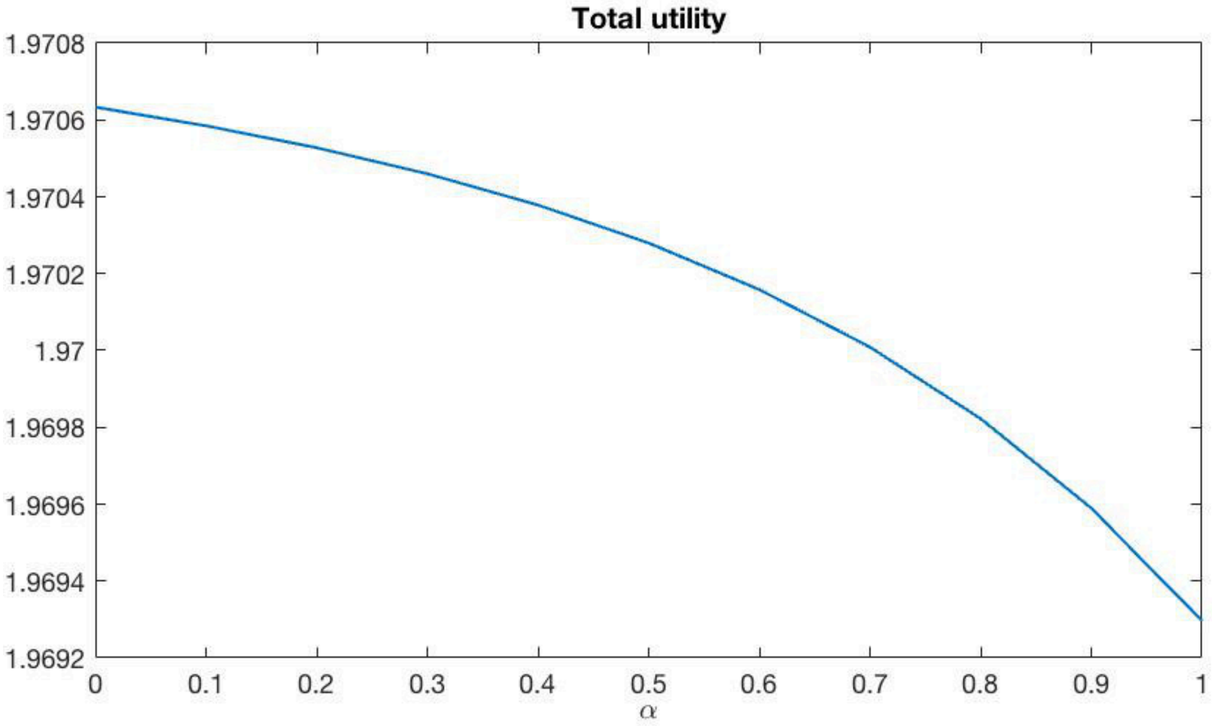


- With the introduction of bequest motive partial annuitisation is optimal.

Numerical simulation: case 2

$$\ddot{a}_x > 1 + e^{-R_f}$$

W_0	β	R_f	i	M	γ	δ	p_x^t	${}^ip_x^t$
1000	0.5	0.1	0.05	0.1	2	0.02	0.5	0.76



Multiperiod model

- For T period:

$$\max_{C,W,0\le\alpha\le1}\sum_{t=0}^{T-1}\left[\frac{C_t^{1-\gamma}-1}{1-\gamma}\frac{p_x^t}{(1+\delta)^t}+\beta\frac{W_{t+1}^{1-\gamma}-1}{1-\gamma}\left(1-p_{x+t}^1\right)\frac{p_x^t}{(1+\delta)^t}\right]$$

$$s.t.\ W_1=\left[W_0-\alpha W_0+\frac{\alpha W_0}{\ddot{a}_x}-C_0\right]\cdot\left[e^{R_f}\right]$$

$$W_{t+1}=(W_t+\frac{\alpha W_0}{\ddot{a}_x}-C_t)\cdot e^{R_f}$$

$$\ddot{a}_x=\frac{1}{1-M}\sum_{t=0}^{T-1}\frac{1}{(1+i)^t}\ {}^{i}p_x^t$$

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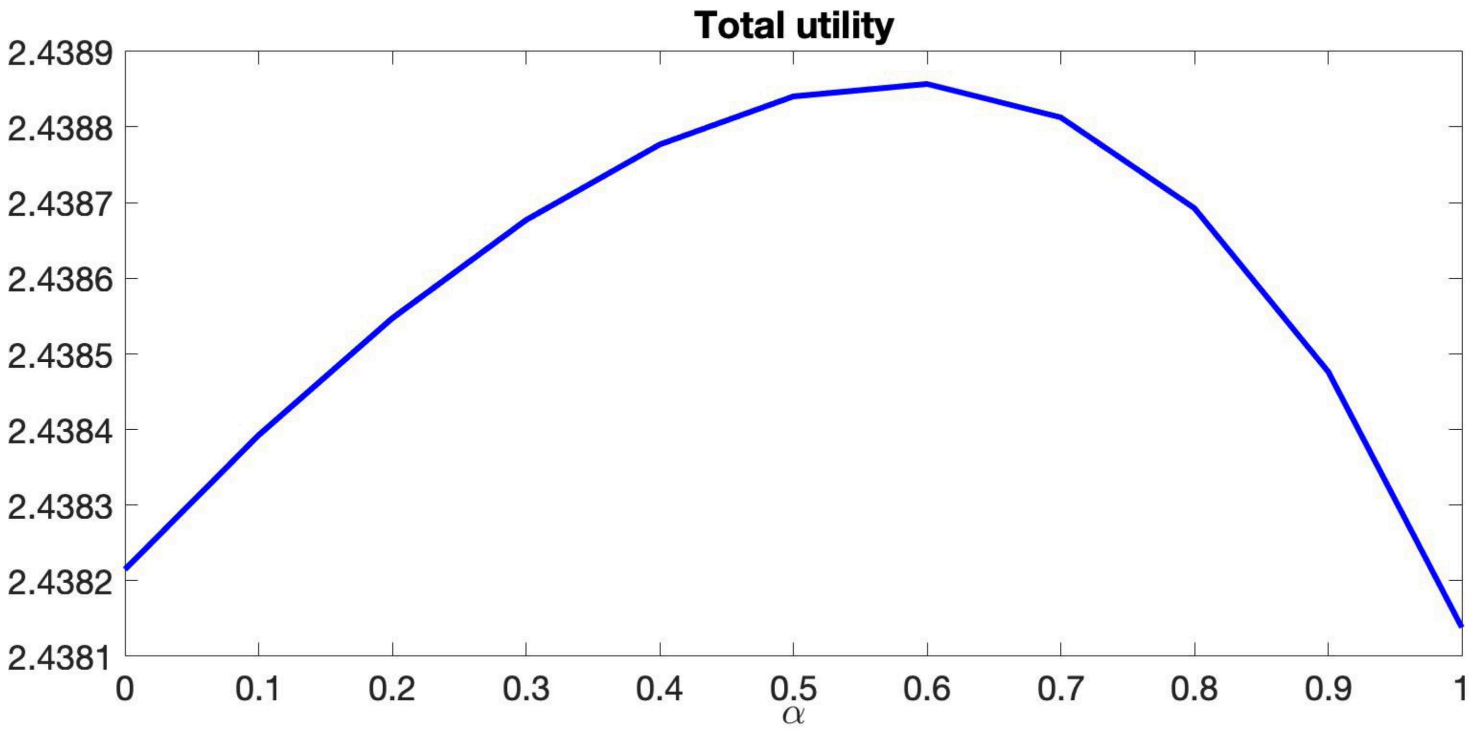
Numerical simulation: case 1

$$\ddot{a}_x < \sum_{t=0}^{T} [e^{R_f}]^{-t}$$

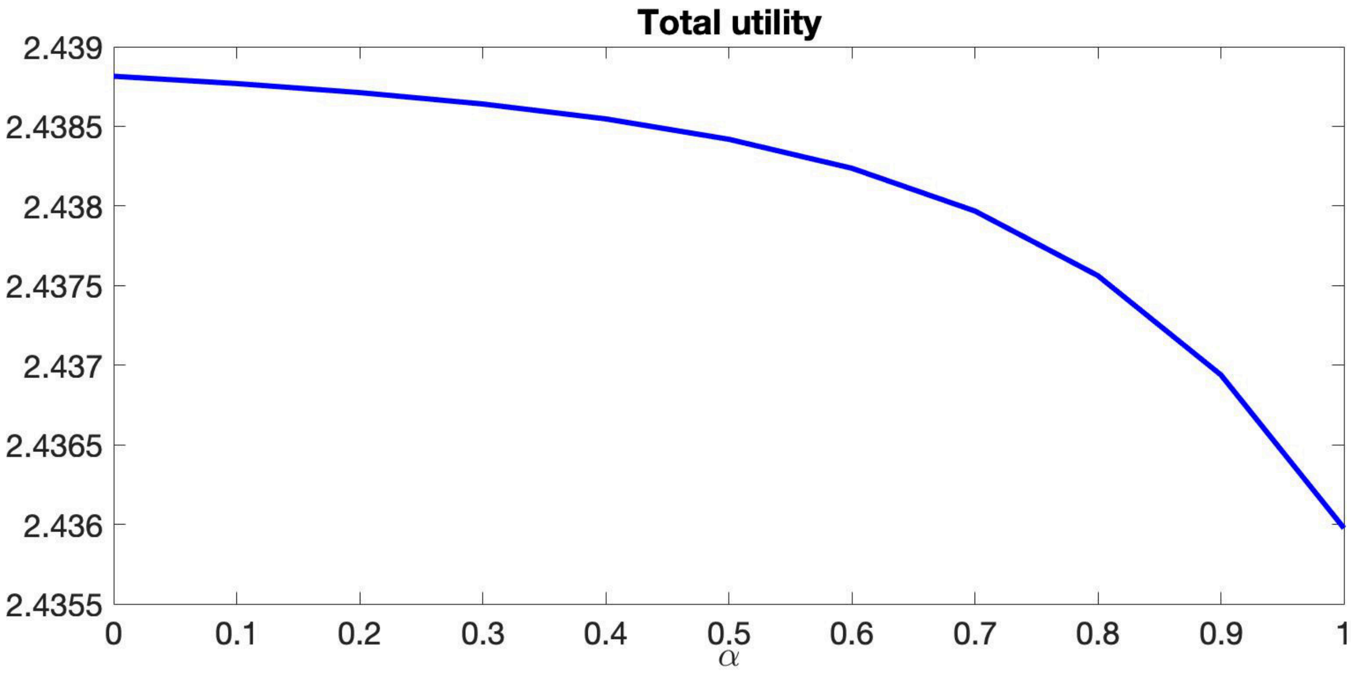
W_0	β	R_f	i	M	γ	δ	p_x^1	p_x^2	${}^i p_x^1$	${}^i p_x^2$
1000	0.5	0.02	0.015	0.05	2	0.02	0.66	0.33	0.7	0.4



Numerical simulation: case 2

$$\ddot{a}_x > \sum_{t=0}^{T} [e^{R_f}]^{-t}$$

W_0	β	R_f	i	M	γ	δ	p_x^1	p_x^2	${}^i p_x^1$	${}^i p_x^2$
1000	0.5	0.12	0.01	0.15	2	0.02	0.66	0.33	0.8	0.5



Further development

- The further extension of the research will be:
- Stochastic financial return.

  Binomial or Brownian motion.
- Multi annuitisation.
  - Possibility to choose every year the fraction to annuitise.

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Thank you!

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