

# The Application of Affine Processes in Cohort Mortality Risk Models

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Zhiping Huang, Michael Sherris, Andrés M. Villegas and Jonathan Ziveyi.

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# Introduction

- Mortality models have attracted research attention over recent years because of the significance of longevity risks (The Joint Forum (2013)) and capital requirements for insurers (Barrieu et al. (2012)).
- Continuous time mortality models (Milevsky and Promislow (2001), Dahl (2004), Cairns et al. (2006a), Blackburn and Sherris (2013)) have received less attention than discrete time mortality models (Lee and Carter (1992), Cairns et al. (2009), Cairns et al. (2006b), Renshaw and Haberman (2006)).
- Continuous time affine cohort mortality models have attracted more recent research (Dahl and Møller (2006), Biffis (2005), Luciano et al. (2008), Schrager (2006), Jevtic et al. (2013), Xu et al. (2015), Chang and Sherris (2018)).

# Research motivation

- We propose an affine mortality model based on the Arbitrage-Free Nelson-Siegel (AFNS) yield curve model (Christensen et al. (2011)) with identifiable factors of level, slope and curvature of the mortality curve.
- We assess a number of continuous-time affine cohort mortality models (Blackburn-Sherris dependent and independent factor mortality models, AFNS independent and dependent factor mortality models, CIR mortality model).
- We investigate the impact of incorporating factor dependence to capture age correlations for the models.

We capture cohort effects directly using age-cohort data to calibrate and assess the model survival curves fit and forecasting performance.

# Affine Mortality Models

- The dynamics of the latent factors  $X_t$  are given by the following system of stochastic differential equations (SDEs) under the risk-neutral measure  $Q$  (Duffie and Kan, 1996; Christensen et al., 2011):

$$dX_t = K^Q [\theta^Q - X_t] dt + \Sigma D(X_t, t) dW_t^Q, \quad (1)$$

where  $K^Q \in \mathbb{R}^{n \times n}$  is the mean reversion matrix,  $\theta^Q \in \mathbb{R}^n$  is the long-term mean,  $\Sigma \in \mathbb{R}^{n \times n}$  is the volatility matrix,  $W_t^Q \in \mathbb{R}^n$  is a standard Brownian motion, and  $D(X_t, t)$  is a diagonal matrix with the  $i$ th diagonal entry as  $\sqrt{\alpha^i(t) + \beta_1^i(t) x_t^1 + \dots + \beta_n^i(t) x_t^n}$ .  $\alpha$  and  $\beta$  are bounded continuous functions.

# Affine Mortality Models

- Under these dynamics the risk-neutral survival probabilities for any age  $x$  from time  $t$  to time  $T$  can be represented as (Blackburn and Sherris, 2013):

$$S(t, T) = \exp\left(B(t, T)' X_t + A(t, T)\right), \quad (2)$$

where  $B(t, T)$  and  $A(t, T)$  are the solutions to the following system of ordinary differential equations (ODEs):

$$\frac{dB(t, T)}{dt} = \rho_1 + (K^Q)' B(t, T), \quad (3)$$

$$\frac{dA(t, T)}{dt} = -B(t, T)' K^Q \theta^Q - \frac{1}{2} \sum_{j=1}^3 \left( \Sigma' B(t, T) B(t, T)' \Sigma \right)_{j,j}, \quad (4)$$

# Multi-Factor Affine Cohort Mortality Models

- The form of risk premium is (Duffee, 2002):

$$\Lambda_t = \begin{cases} \lambda^0 + \lambda^1 X_t, & \text{models with Gaussian processes;} \\ D(X_t, t) \lambda^0, & \text{the CIR model,} \end{cases} \quad (5)$$

where  $\Lambda_t \in \mathbb{R}^{n \times 1}$ ,  $\lambda^0 \in \mathbb{R}^{n \times 1}$  and  $\lambda^1 \in \mathbb{R}^{n \times n}$ .

- With these assumptions, the SDEs for factors under the measure  $P$  can be written as:

$$dX_t = \begin{cases} K^P [\theta^P - X_t] dt + \Sigma dW_t^P, & \text{models with Gaussian processes;} \\ K^P [\theta^P - X_t] dt + \Sigma D(X_t, t) dW_t^P, & \text{the CIR model,} \end{cases} \quad (6)$$

# Multi-Factor Affine Cohort Mortality Models

The dynamics of the factors in each model that we will estimate from historical mortality data is:

- The independent Blackburn-Sherris model (Blackburn and Sherris, 2013)

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} = - \begin{pmatrix} \delta_{11} & 0 & 0 \\ 0 & \delta_{22} & 0 \\ 0 & 0 & \delta_{33} \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \begin{pmatrix} dW_t^{1,Q} \\ dW_t^{2,Q} \\ dW_t^{3,Q} \end{pmatrix}. \quad (7)$$

- The independent AFNS model (Christensen et al., 2011) The dynamics of the factors under the  $Q$ -measure are given by:

$$\begin{pmatrix} dL_t \\ dS_t \\ dC_t \end{pmatrix} = - \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta & -\delta \\ 0 & 0 & \delta \end{pmatrix} \begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \begin{pmatrix} dW_t^{1,Q} \\ dW_t^{2,Q} \\ dW_t^{3,Q} \end{pmatrix}. \quad (8)$$

- The dependent Blackburn-Sherris model

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} = - \begin{pmatrix} k_{11}^P & 0 & 0 \\ 0 & k_{22}^P & 0 \\ 0 & 0 & k_{33}^P \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ \sigma_{31} & \sigma_{32} & 0 \end{pmatrix} \begin{pmatrix} dW_t^{1,P} \\ dW_t^{2,P} \\ dW_t^{3,P} \end{pmatrix}.$$

# Multi-Factor Affine Cohort Mortality Models

- The dependent AFNS model

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} = - \begin{pmatrix} \delta_{11} & 0 & 0 \\ \delta_{21} & \delta_{22} & 0 \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} dW_t^{1,Q} \\ dW_t^{2,Q} \\ dW_t^{3,Q} \end{pmatrix}. \quad (10)$$

- The CIR model

$$\begin{aligned} \begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} &= - \begin{pmatrix} \delta_{11} & 0 & 0 \\ 0 & \delta_{22} & 0 \\ 0 & 0 & \delta_{33} \end{pmatrix} \left[ \begin{pmatrix} \theta_1^Q \\ \theta_2^Q \\ \theta_3^Q \end{pmatrix} - \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} \right] dt \\ &+ \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \begin{pmatrix} \sqrt{X_t^1} & 0 & 0 \\ 0 & \sqrt{X_t^2} & 0 \\ 0 & 0 & \sqrt{X_t^3} \end{pmatrix} \begin{pmatrix} dW_t^{1,Q} \\ dW_t^{2,Q} \\ dW_t^{3,Q} \end{pmatrix}. \end{aligned} \quad (11)$$



# Multi-Factor Affine Cohort Mortality Models

$\tau = T - t$  times  $\bar{\mu}^i(x; t, T)$  is given by  $-[B(t, T)' X_t + A(t, T)]$  where  $B(t, T)$ , the factor loadings, and  $A(t, T)$  have explicit expressions.

- The independent Blackburn-Sherris model (Blackburn and Sherris, 2013)

$$B^j(t, T) = -\frac{1 - e^{-\delta_{jj}(T-t)}}{\delta_{jj}}, \quad j = 1, 2, 3, \quad (12)$$

$$A(t, T) = \frac{1}{2} \sum_{j=1}^3 \frac{\sigma_{jj}^2}{\delta_{jj}^3} \left[ \frac{1}{2} \left( 1 - e^{-2\delta_{jj}(T-t)} \right) - 2 \left( 1 - e^{-\delta_{jj}(T-t)} \right) + \delta_{jj}(T-t) \right]. \quad (13)$$

# Multi-Factor Affine Cohort Mortality Models

- The independent AFNS model (Christensen et al., 2011)

$$B^1(t, T) = -(T - t), \quad B^2(t, T) = -\frac{1 - e^{-\delta(T-t)}}{\delta}, \quad (14)$$

$$B^3(t, T) = (T - t) e^{-\delta(T-t)} - \frac{1 - e^{-\delta(T-t)}}{\delta},$$

$$\frac{A(t, T)}{T - t} = \sigma_{11}^2 \frac{(T - t)}{6} + \sigma_{22}^2 \left[ \frac{1}{2\delta^2} - \frac{1}{\delta^3} \frac{1 - e^{-\delta(T-t)}}{T - t} + \frac{1}{4\delta^3} \frac{1 - e^{-2\delta(T-t)}}{T - t} \right] +$$

$$\sigma_{33}^2 \left[ \frac{1}{2\delta^2} + \frac{1}{\delta^2} e^{-\delta(T-t)} - \frac{1}{4\delta} (T - t) e^{-2\delta(T-t)} - \frac{3}{4\delta^2} e^{-2\delta(T-t)} \right.$$

$$\left. - \frac{2}{\delta^3} \frac{1 - e^{-\delta(T-t)}}{T - t} + \frac{5}{8\delta^3} \frac{1 - e^{-2\delta(T-t)}}{T - t} \right]$$

# Mortality Data

- US mortality age-cohort data from the Human Mortality Database (2017) (HMD) to calibrate and compare the mortality models.
  - Mortality data of males from ages 50 to 100 for the cohorts born from 1883 to 1915.
- Historical survival probability,  $S^i(x; t, T)$ , and the historical average forces of mortality  $\bar{\mu}^i(x; t, T)$  over the period  $\tau = T - t$  for each cohort  $i$  aged  $x$  at time  $t$  from the data, using:

$$S^i(x; t, T) = \prod_{s=1}^{T-t} [1 - q^i(x + s - 1, t + s - 1)], \quad (16)$$

$$\bar{\mu}^i(x; t, T) = -\frac{1}{T-t} \log [S^i(x; t, T)], \quad (17)$$

where  $q^i(x, t)$  is the one year death probability for an individual aged  $x$  at time  $t$  in cohort  $i$ .

# Mortality Data

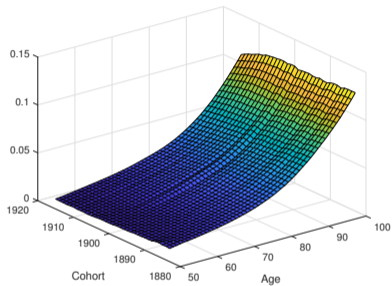


Figure 1: Average Force of Mortality for Males Born from 1883 to 1915

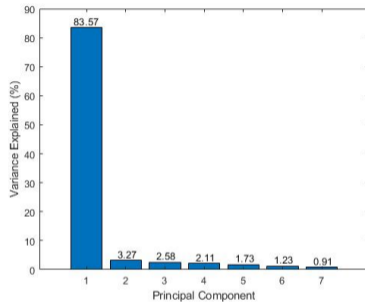


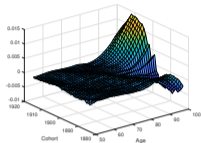
Figure 2: Fractions (%) of Variance Explained by Each of the First 7 Principal Components

# Model Goodness of Fit

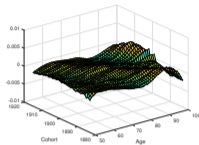
Table 1: Comparison of Affine Mortality Models

	The Blackburn-Sherris Model		The AFNS Model		The CIR Model
	Independent-Factor	Dependent-Factor	Independent-Factor	Dependent-Factor	
Log Likelihood	9896.419	9938.696	9665.801	9887.878	10045.70
RMSE	0.00250	7.601e-04	6.856e-04	9.160e-04	5.227e-04
No. of Parameters	12	18	10	13	18
AIC	-19570.837	-19643.392	-19113.602	-19551.757	-19857.40
BIC	-18968.292	-19008.277	-18521.914	-18943.783	-19222.29
Probability of Negative Mortality	0.02700	1.011e-32	1.722e-31	4.34e-14	-

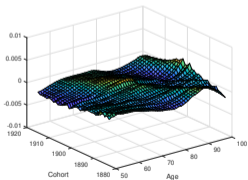
# Model Residual Analysis



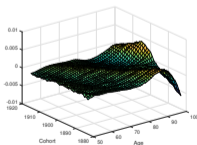
(a) The Independent  
Blackburn-Sherris Model



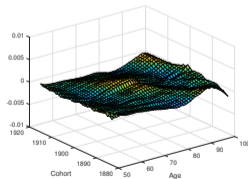
(b) The Dependent  
Blackburn-Sherris Model



(c) The Independent AFNS  
Model

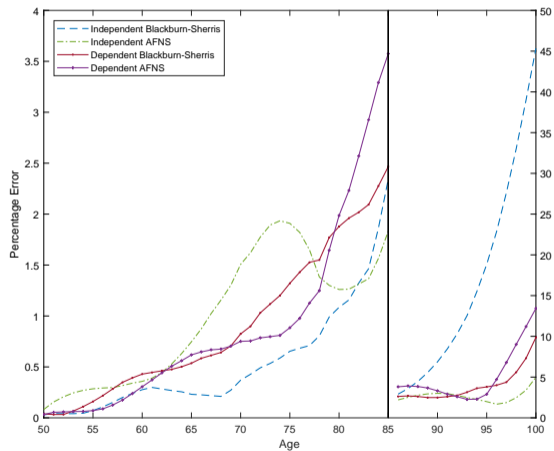


(d) The Dependent AFNS  
Model



(e) The CIR Model

# MAPE of Affine Mortality Models



# MAPE of Affine Mortality Models

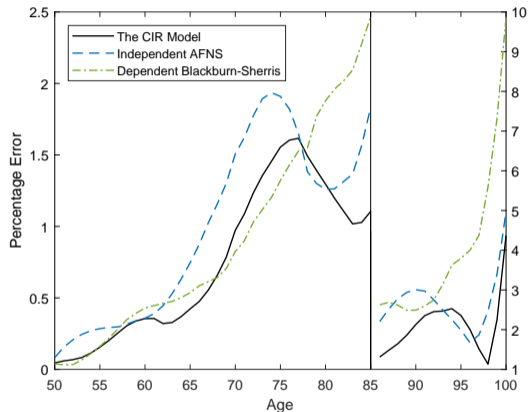


Figure 5: The CIR Model, the Dependent Blackburn-Sherris Model and the Independent AFNS Model



# Forecast RMSE

Table 2: RMSE by Comparing the Actual and Best-Estimate Survival Probabilities of the 1916 Cohort

	The Blackburn-Sherris Model		The AFNS Model		The CIR Model
	Independent	Dependent	Independent	Dependent	
RMSE	0.03197	0.00726	0.00668	0.00754	0.01835

# Forecast RMSE

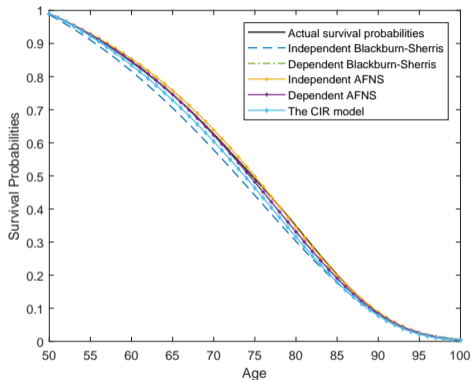


Figure 6: Actual and Best-Estimate survival Probabilities of the 1916 Cohort

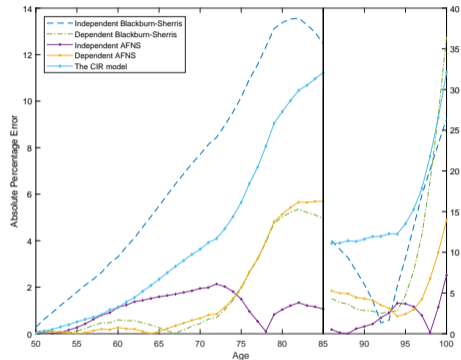


Figure 7: Absolute Percentage Errors between Actual and Best-Estimate Survival Probabilities

# Summary and Conclusions

- We introduce an AFNS mortality model with interpretable latent stochastic factors for level, slope and curvature of the survival curve.
- Incorporating dependence in the the Blackburn-Sherris mortality model improves in-sample model fit and out-of-sample forecasting performance.
- The CIR mortality model has the best in-sample model fit reflecting the more realistic assumption of Gamma-distributed mortality rates.
- Independent-factor AFNS mortality model is parsimonious, can better capture the variation in cohort mortality rates in US data, a better fit at older ages than the independent-factor Blackburn-Sherris model, better predictive performance. Negative mortality rates have very low probability. Intuitive factor interpretation and well suited for financial and insurance applications.

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